

Destruction or Preservation As You Like It

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Abstract. The Gap Forcing Theorem, a key contribution of this paper, implies essentially that after any reverse Easton iteration of closed forcing, such as the Laver preparation, every supercompactness measure on a supercompact cardinal extends a measure from the ground model. Thus, such forcing can create no new supercompact cardinals, and, if the GCH holds, neither can it increase the degree of supercompactness of any cardinal; in particular, it can create no new measurable cardinals. In a crescendo of what I call exact preservation theorems, I use this new technology to perform a kind of partial Laver preparation, and thereby finely control the class of posets which preserve a supercompact cardinal. Eventually, I prove the ‘As You Like It’ Theorem, which asserts that the class of $<\kappa$ -directed closed posets which preserve a supercompact cardinal κ can be made by forcing to conform with any pre-selected local definition which respects the equivalence of forcing. Along the way I separate completely the levels of the superdestructibility hierarchy, and, in an epilogue, prove that the notions of fragility and superdestructibility are orthogonal—all four combinations are possible.

There is a vast unknown continent, which I aim to explore, between two extreme theorems: Laver [Lav78], on the one hand, proved, in what is perhaps my favorite argument in large cardinal set theory, that a supercompact cardinal can be made *indestructible*, so that its supercompactness is preserved by every $<\kappa$ -directed closed forcing notion; in my Superdestruction Theorem [Ham97b], on the other hand, and in joint work with Shelah [HS97], I proved that a supercompact cardinal can be made *superdestructible*, so that its supercompactness is destroyed by every $<\kappa$ -closed forcing notion. Are there any theorems in the uncharted wilderness between these two extremes? Indeed there are, and in this paper I will prove that there are. Here, between indestructibility and superdestructibility, I will lean alternately on the methods of [Lav78] when I want a poset to preserve supercompactness and then on the Gap Forcing Theorem, introduced in this paper, when I want it to destroy supercompactness. In a crescendo of Exact Preservation theorems, my results will culminate in the ‘As You Like It’ Theorem, which asserts that the class of $<\kappa$ -directed closed posets which preserve the supercompactness of κ can be made to conform with any pre-selected local definition which respects the equivalence of forcing. This theorem and the others I prove like it fill the region between indestructibility and superdestructibility. I hope that my techniques will allow you to prove that the class of posets which preserve the supercompactness of κ can be made to be the class you have always wished that it was, whatever that may be.

I confess that this paper is inside-out. After proving some preliminary facts, I march through a sequence of exact preservation theorems, the later ones stronger

versions of earlier ones, finally advancing to the As You Like It Theorem. In an outside-out fashion, I could simply have proved the As You Like It Theorem first, and then deduced the earlier theorems as corollaries. But I chose this backwards manner of presentation because I view the main contribution of this paper as the *method of proof* of these theorems. So, in order to highlight the power of this method, I start slowly and then build up to the stronger theorems. I hope that the logical overlap that this order of presentation involves will be forgiven.

Let me explain the paper's overall structure. I begin in section one with my main new tool, the Gap Forcing Theorem. In sections two and three I introduce the useful concepts of a partial Laver preparation and a high jump function, respectively, before giving my first applications in section four: the Exact Preservation Theorems. In section five, in order to improve these theorems, I develop the theory of representability, and, in section six, apply it to separate the levels of the superdestructibility hierarchy. In section seven, I present much more powerful Exact Preservation Theorems, and culminate in the 'As You Like It' Theorem, the title theorem of this paper. Finally, in an epilogue, I separate the notions of superdestructibility and fragility.

I try whenever possible to use standard notation, but assume a familiarity with reverse Easton forcing iterations, such as the Laver preparation, and the lifting arguments they typically involve. Following Adrian Mathias, I use the notation $h : X \rightarrow Y$ to mean that h is a partial function from X to Y .

§1 Gap Forcing

The key new technology in this paper is the Gap Forcing Theorem and its corollaries, which give explicit information about the nature of supercompactness embeddings which are added by forcing. Since they severely limit the kinds of measures which can exist after gap forcing, I will use them as a fundamental tool when proving the Exact Preservation Theorems, where I must make certain forcing notions destroy supercompactness.

The Gap Forcing Theorem will depend crucially on an improved version of The Key Lemma, below, which was proved initially in [Ham97b] but was also instrumental in the main results of [HamSh1]. As in [Ham97b], define that a sequence or a set of ordinals is *fresh* over V when it is not in V , but every proper initial segment of it is in V .

Key Lemma 1.1 *If $|\mathbb{P}| = \beta$, $\text{cof}(\lambda) > \beta$, and $\Vdash_{\mathbb{P}} \mathbb{Q}$ is $\leq \beta$ -closed, then $\mathbb{P} * \mathbb{Q}$ adds no fresh subsets of λ .*

In this paper I will use an improved version of the Key Lemma. Before doing so, let me define that a poset is $\leq \beta$ -strategically closed when the second player has a winning strategy in the game of length β in which the players create a descending sequence $\langle p_\alpha \mid \alpha < \beta \rangle$ from the poset, the second player playing at every limit stage. The first player to violate the rule that the conditions descend loses, and otherwise the second player wins.

Improved Key Lemma 1.2 *If $|\mathbb{P}| = \beta$, $\text{cof}(\lambda) > \beta$, and $\Vdash_{\mathbb{P}} \mathbb{Q}$ is $\leq \beta$ -strategically closed, then $\mathbb{P} * \mathbb{Q}$ adds no fresh subsets of λ , and no fresh λ -sequences.*

Proof: There are two improvements. First, the original proof of the Key Lemma in [Ham97b] works in the case, here, of strategically closed \mathbb{Q} , just as well: one simply needs to take care, when defining q_t in that argument, to also obey the strategy, so that the limit stages will go through. Second, the improvement to λ -sequences is obtained by coding elements of δ^λ with binary sequences of length $\delta \cdot \lambda$, which has the same cofinality as λ . \square_{Lemma}

Main Definition 1.3 A forcing notion \mathbb{P} admits a *gap* at δ when the poset \mathbb{P} factors as $\mathbb{P}_1 * \mathbb{P}_2$ where $|\mathbb{P}_1| < \delta$ and \mathbb{P}_2 is $\leq \delta$ -strategically closed in $V^{\mathbb{P}_1}$. The Laver preparation, and indeed most every reverse Easton iteration of closed forcing, admits a gap between any two stages of the forcing.

Definition 1.4 A set C is *unbounded* in $P_\kappa \gamma$ when for every $\sigma \in P_\kappa \gamma$ there is $\tau \in C$ with $\sigma \subseteq \tau$. A set $D \subseteq C$ is δ -directed if for any set $B \subseteq D$ of size less than δ there is $\sigma \in D$ such that $\cup B \subseteq \sigma$. The set C is δ -closed when every δ -directed $D \subseteq C$ with $|D| < \kappa$ has $\cup D \in C$. Define that C is δ -club when it is unbounded and δ -closed (Caution: this usage differs from that in, say, [Kan94]). A *supercompactness* measure is simply a normal fine measure on $P_\kappa \lambda$ for some λ .

I view the next theorem, evolved from lesser forms into the highly useful current animal, as a full-grown version of the Key Lemma. It and its corollaries will provide the essential new tools with which I will deduce that certain kinds of forcing notions destroy supercompactness.

You may underappreciate this theorem if you are not familiar with the bizarre sorts of embeddings which can live in a forcing extension. It is easy, for example, to construct a forcing extension $V[G]$ with an embedding $j : V[G] \rightarrow M[j(G)]$ such that $M \not\subseteq V$. Indeed, M may even have different subsets of $\kappa = \text{cp}(j)$ than V . Adding a Cohen subset to a supercompact cardinal κ which is indestructible in V ,

for example, of necessity produces such embeddings, since the new subset $G \subseteq \kappa$ must be in $M[j(G)]$ but cannot have been added by $j(G)$, and so it is in M but not in V . Of course, however, a standard technique to show supercompactness is preserved to a forcing extension is to lift an embedding $j : V \rightarrow M$ from the ground model V to the extension $j : V[G] \rightarrow M[j(G)]$, and these embeddings have the properties listed in the conclusion of the Gap Forcing Theorem. But that is exactly the force and utility of the theorem—the amazing fact the Gap Forcing Theorem and its corollaries identify is that after forcing which admits a gap below κ , *every* supercompactness embedding resembles the lift of a supercompactness embedding in V . But this is only resemblance: we cannot prove that every λ -supercompactness measure in a forcing extension $V[G]$ which admits a gap below κ is the lift of a λ -supercompactness measure in V . This is because, as I proved in my dissertation [Ham94b], a strong embedding in V , which is not an ultrapower embedding at all, can be lifted via gap forcing to an embedding which in the extension is the ultrapower by a normal measure on κ . So we should perhaps be content to know that the embeddings in $V[G]$ resemble lifts as much as the claims made in the Gap Forcing Theorem require them to. Of course, I am speaking here of embeddings which are internal to $V[G]$ in the sense that they are definable there.

Gap Forcing Theorem 1.5 *Suppose that $V[G]$ is a forcing extension of V which admits a gap below κ , and that $j : V[G] \rightarrow M[j(G)]$ is an embedding defined in $V[G]$, with critical point κ , which is closed under λ -sequences, i.e. $M[j(G)]^\lambda \subseteq M[j(G)]$ in $V[G]$, for some λ with $\kappa \leq \lambda < j(\kappa)$. Then:*

1. $M \subseteq V$,
2. $(M^\lambda)^M = (M^\lambda)^V$,
3. $j \restriction \lambda \in M$, and
4. $j \restriction \lambda^{+(n)V} \in V$ for every $n \in \omega$. (This can be pushed higher depending on the distributivity of the forcing.)

Proof: Suppose that $G = g * H \subseteq \mathbb{P} * \mathbb{Q}$ exhibits the gap at $\delta < \kappa$, so that $|\mathbb{P}| < \delta$ and \mathbb{Q} is $\leq \delta$ -strategically closed in $V[g]$. Without loss of generality I may assume that $\mathbb{P} \in V_\kappa$, so that $j(\mathbb{P}) = \mathbb{P}$.

Before proving the claims, let me first establish the preliminary claim that $M^\lambda \cap V = V^\lambda \cap M$. It suffices to show by induction on $\gamma \leq \lambda$ that $M^\gamma \cap V = V^\gamma \cap M$. Suppose this is true up to γ , and consider now the two directions to prove at γ . Suppose, for (\subseteq) , that $s \in M^\gamma \cap V$. Thus, $s : \gamma \rightarrow M$ and $s \in V$; I aim to show $s \in M$. By the induction hypothesis every initial segment of s is in M . Certainly

$s \in M[j(G)]$, by the closure assumption on j . If $\text{cof}(\gamma) > \delta$, then s must in fact be in M , for otherwise it would be an M -fresh sequence added by the gap forcing $j(\mathbb{P} * \mathbb{Q})$, contrary to the Key Lemma 1.2. So assume that $\text{cof}(\gamma) \leq \delta$. Thus, by the closure of $j(\mathbb{Q})$, it follows that $s \in M[g]$, and so $s = \dot{s}_g$ for some name $\dot{s} \in M$. View \dot{s} as a function from γ to the set of antichains in \mathbb{P} labelled with the possible elements of $\text{ran}(s)$, which are all in both M and V . By the closure of \mathbb{Q} , it follows that $\dot{s} \in V[g]$, and so $\dot{s} = \ddot{s}_g$ for some name-of-a-name $\ddot{s} \in V$. In $V[g]$ we have $(\ddot{s}_g)_g = s$, and so this is forced by some condition $p \in g$. The condition p must decide enough information about the name-of-a-name \ddot{s} to determine that it is a name which p decides agrees with the sequence s . (This is an unusual use of a name-of-a-name in that unlike iterated forcing, here I am using the *same* generic twice to interpret the name in $(\ddot{s}_g)_g$.) Whatever information p decides about \ddot{s} must agree with \dot{s} , since $\dot{s} = \ddot{s}_g$, so it follows that from the condition p and \dot{s} in M I can construct s . So $s \in M$.

For the converse direction, suppose $s : \gamma \rightarrow V$ and $s \in M$. I aim to show that $s \in V$. By the induction hypothesis, every initial segment of s is in V , and so if $\text{cof}(\gamma) > \delta$, the Key Lemma 1.2 yields $s \in V$. So assume again that $\text{cof}(\gamma) \leq \delta$. Thus $s \in V[g]$ by the closure of \mathbb{Q} , and consequently $s = \dot{s}_g$ for some name $\dot{s} \in V$. By coding the name \dot{s} as a γ -sequence, it follows by the previous direction that $\dot{s} \in M$. Thus, both s and \dot{s} are in M , and in $M[g]$ we have $s = \dot{s}_g$. Thus, there is a condition $p \in g$ forcing this. Using p and \dot{s} in V , I can construct s . So $s \in V$, as desired. This proves the preliminary claim.

Now let me prove 1. It suffices to show that $P(\beta)^M \subseteq V$ for every ordinal β . Suppose $X \in P(\beta)^M$ and, by induction, every initial segment of X is in V . I aim to show $X \in V$. If $\text{cof}(\beta) > \delta$, then X is in V by the Key Lemma 1.1. So assume, alternatively, that $\text{cof}(\beta) \leq \delta$, and consequently that $\beta = \sup_{\alpha < \delta} \beta_\alpha$ in M , with $\beta_\alpha < \beta$. In this case $\langle X \cap \beta_\alpha \mid \alpha < \delta \rangle$ is in $V^\delta \cap M$, and consequently it is in V by the preliminary claim. By taking a union it follows that $X \in V$ as well. This establishes 1. And 2 follows immediately from 1 and the preliminary claim.

To prove 3, it suffices to show by induction on γ that $j \restriction \gamma \in M$ for all $\gamma \leq \lambda$. Assume by induction that this claim holds up to γ , and consider $j \restriction \gamma$. I may assume $\gamma > \kappa$. Note that $j \restriction \lambda \in M[j(G)]$, by the assumption on j . In the case that $\text{cof}(\gamma) > \delta$, the Key Lemma 1.2 implies $j \restriction \gamma \in M$ since otherwise it would be an M -fresh sequence, and these cannot be added by the gap forcing $j(\mathbb{P} * \mathbb{Q})$. So I may assume $\text{cof}(\gamma) \leq \delta$ and consequently $j \restriction \gamma \in M[g]$. Let $C = (P_\kappa \gamma)^V$. By 2 this

is equal to $(P_\kappa\gamma)^M$. Consequently, $j \restriction C = \{j(\sigma) \mid \sigma \in C\} = \{j \restriction \sigma \mid \sigma \in (P_\kappa\gamma)^M\}$, and so $j \restriction C$ is constructible from $j \restriction \gamma$ in $M[g]$.

Let me now argue that C is a δ -club in $V[g]$. This part of the argument is similar to [HamShl]. First, let me argue that C is unbounded. Any $\sigma \in (P_\kappa\gamma)^{V[g]}$ comes from a name $\dot{\sigma}$, and is therefore covered by $\tau = \{\alpha \mid \llbracket \dot{\alpha} \in \dot{\sigma} \rrbracket \neq 0\}$. It must be that $\tau \in C$ since \mathbb{P} is small. Second, let me show that C is δ -closed. Suppose that $D \subseteq C$ has size less than κ and is δ -directed in $V[g]$. I want to show that $\cup D \in C$. It suffices to show that $\cup D \in V$. Let \dot{D} be a name for D , and let $D_p = \{\sigma \in C \mid p \Vdash \dot{\sigma} \in \dot{D}\}$. Thus, $D = \cup_{p \in g} D_p$. There must be some $p \in g$ such that D_p is \subseteq -cofinal in D for if not, then for each $p \in g$ I may choose $\sigma_p \in D$ such that D_p contains no supersets of σ_p . Since D is δ -directed and $|g| < \delta$ there must be a $\sigma \in D$ such that $\sigma_p \subseteq \sigma$ for each $p \in g$. But σ must be forced in D by some condition $p \in g$, so $\sigma \in D_p$ for some $p \in g$, contrary to the choice of σ_p . Thus, there is some $p \in g$ such that D_p is \subseteq -cofinal in D , and consequently $\cup D = \cup D_p \in V$, as desired. So C is δ -club in $V[g]$.

Therefore $j(C)$ is a δ -club in $M[g]$. Also, $j \restriction C \in M[g]$. But $j \restriction C$ is δ -directed by \subseteq and has size less than $j(\kappa)$. It follows that $\cup(j \restriction C) \in j(C)$. But $\cup(j \restriction C) = \cup\{j(\sigma) \mid \sigma \in C\} = \cup\{j \restriction \sigma \mid \sigma \in C\} = j \restriction \gamma$, and so I conclude that $j \restriction \gamma \in j(C) \in M$. Thus, $j \restriction \gamma \in M$, as desired. At the top of the induction, I conclude $j \restriction \lambda \in M$. So 3 holds.

Now let me prove 4. By 1 and 3, it follows that $j \restriction \lambda \in V$. Suppose now that $j \restriction \gamma \in V$, where $\lambda \leq \gamma$; I will argue that $j \restriction \gamma^+ \in V$ as well (meaning $(\gamma^+)^V$). A simple induction on n will then prove the theorem. By the Key Lemma 1.2 it suffices to show that every initial segment of $j \restriction \gamma^+$ is in V . So suppose $\gamma \leq \beta < \gamma^+$. There must be in V a relation \triangleleft on γ such that $\beta = \text{ot}\langle \gamma, \triangleleft \rangle$. But if α has order-type $\zeta < \beta$ with respect to \triangleleft , then $j(\alpha)$ will have order-type $j(\zeta)$ with respect to $j(\triangleleft)$. Consequently

$$j \restriction \beta = \{\text{order-type of } j(\alpha) \text{ wrt } j(\triangleleft) \mid \alpha < \gamma\},$$

which is constructible from $j(\triangleleft)$ and $j \restriction \gamma$ in V , as desired. Lastly, in the event that \mathbb{P} is $<\beta$ -distributive, one easily obtains the limit stages and I conclude that $j \restriction \lambda^{+(\alpha)} \in V$ for every $\alpha < \beta$. \square_{Theorem}

Please observe that I did not use the hypothesis that $\lambda < j(\kappa)$ when proving 1 and 2, so that, for example, those conclusions follow when j is a huge embedding in $V[G]$. Also, let me point out one important consequence of the Gap Forcing Theorem. Namely, if $j : V[G] \rightarrow M[j(G)]$ is the λ -supercompactness embedding by

a normal fine measure η on $(P_\kappa\lambda)^{V[G]}$, then actually η concentrates on $(P_\kappa\lambda)^V$, and the reason for this is that $X \in \eta \leftrightarrow j''\lambda \in j(X)$, but $j''\lambda \in M$, so $j''\lambda \in P_{j(\kappa)}j(\lambda)^M = j(P_\kappa\lambda^V)$. So every supercompactness measure added by gap forcing concentrates on a set in the ground model. But actually, as I will now prove, much much more is true:

Gap Forcing Corollary 1.6 *Suppose that κ is supercompact in $V[G]$, a forcing extension which admits a gap below κ . Then κ is supercompact in V , and every supercompactness measure in $V[G]$ extends a measure in V .*

Let me point out that this corollary is a global claim, in the sense that the *full* supercompactness of κ in $V[G]$ is used to obtain the *full* supercompactness of κ in V ; and the source of this phenomenon is identified: every supercompactness measure in $V[G]$ extends a supercompactness measure in V . I will prove the corollary by proving the following more local version, in which a bit more than the λ -supercompactness of κ in $V[G]$ is used to obtain the λ -supercompactness of κ in V . The *finite cardinal successors* of λ are simply the cardinals λ^+ , λ^{++} , λ^{+++} , and so on, obtained by applying the successor operation finitely many times.

Local Improvements 1.7 *If $V[G]$ admits a gap below κ , then:*

1. *If κ is $2^{\lambda^{<\kappa}}$ -supercompact in $V[G]$ then every λ -supercompactness measure in $V[G]$ extends a measure in V .*
2. *If the GCH holds in V , or indeed, if only $2^{\lambda^{<\kappa}}$ is a finite cardinal successor cardinal of λ in V , then again every λ -supercompactness measure in $V[G]$ extends a measure in V .*

Proof: The Gap Forcing Corollary follows from 1, so let me begin with that. Suppose ν is a λ -supercompactness measure in $V[G]$. Let $\theta = 2^{\lambda^{<\kappa}}$, and suppose that κ is θ -supercompact in $V[G]$. Let $j_\mu : V[G] \rightarrow M_\mu[j_\mu(G)]$ be the ultrapower by a θ -supercompactness measure μ in $V[G]$. Since $|\nu| = \theta$, it follows that $j_\mu''\nu \in M_\mu[j_\mu(G)]$, and, since this is a subset of $j_\mu(\nu)$ of size less than $j_\mu(\kappa)$, it follows that it has nonempty intersection. Pick any $s \in \cap j''\nu$. It follows that $X \in \nu \leftrightarrow s \in j_\mu(X)$ for $X \subseteq P_\kappa\lambda$. I will use the Gap Forcing Theorem 1.5 applied to j_μ . Since, by the remarks after the Gap Forcing Theorem, ν concentrates on $(P_\kappa\lambda)^V$, it follows that $s \in M_\mu$, and consequently, $s \in V$. Now let $\langle X_\alpha \mid \alpha < \theta \rangle$ enumerate $P(P_\kappa\lambda)^V$ in V . Thus, $j_\mu(\langle X_\alpha \mid \alpha < \theta \rangle) \in M_\mu$, and by 1, 2, and 3 of the Gap Forcing Theorem applied to j_μ , I may conclude that $\langle j_\mu(X_\alpha) \mid \alpha < \theta \rangle \in V$. Thus, in V I can simply check whether $s \in j_\mu(X_\alpha)$ to know whether $X_\alpha \in \nu$. So $\nu \cap V \in V$. It is easy to verify that $\nu \cap V$ is a normal fine measure on $P_\kappa\lambda$ in V . So ν extends a supercompactness

measure in V . So 1 is proved.

To prove 2, I may not assume that κ is anything more than λ -supercompact in $V[G]$, with the measure ν . But I may assume instead that $\theta = (2^{\lambda^{<\kappa}})^V$ is a finite successor cardinal to λ in V . Suppose $j : V[G] \rightarrow M[j(G)]$ is the ultrapower by ν . By 4 of the Gap Forcing Theorem, $j \restriction \theta \in V$, and I can proceed as before: let $\langle X_\alpha \mid \alpha < \theta \rangle$ enumerate $P(P_\kappa \lambda)^V$ in V . Since $M \subseteq V$, I know that $j(\langle X_\alpha \mid \alpha < \theta \rangle) \in V$. By restricting to $j \restriction \theta$ and collapsing the domain, it follows that $\langle j(X_\alpha) \mid \alpha < \theta \rangle \in V$. Since $X \in \nu \leftrightarrow j \restriction \lambda \in j(X)$, I can in V determine which X_α satisfy $j \restriction \lambda \in j(X_\alpha)$. Thus, $\nu \cap V \in V$, and again ν extends a measure in V , as required. $\square_{\text{Corollary}}$

Just as in the Gap Forcing Theorem 1.5, the GCH assumption in 2 can be relaxed even further when the forcing is distributive. But the natural question remains, whether in the absence of additional strength assumed in $V[G]$ or GCH-type hypotheses in V , the completely local result holds: that is, in a gap forcing extension $V[G]$, must every cardinal κ which is λ -supercompact in $V[G]$ be λ -supercompact in V ? While I don't know the answer, my subsequent results need only the hypotheses stated here.

Corollary 1.8 *If κ is supercompact in $V[G]$, a forcing extension which admits a gap below κ , then every κ -complete measure in $V[G]$ which concentrates on a set in V extends a measure in V .*

Proof: Suppose that ν is a κ -complete measure in $V[G]$ which concentrates on a set $D \in V$. Let $\lambda = |D|$, and let $j : V[G] \rightarrow M[j(G)]$ be a 2^λ -supercompactness embedding in $V[G]$. Thus, $j \restriction 2^\lambda \in V$ and $M \subseteq V$ by the Gap Forcing Theorem. It follows that if $\langle X_\alpha \mid \alpha < 2^\lambda \rangle$ enumerates $P(D)$ in V , then $\langle j(X_\alpha) \mid \alpha < 2^\lambda \rangle \in V$. Since j is 2^λ -supercompact, it follows that $j \restriction \nu \in M[j(G)]$, and therefore, since $j(\nu)$ is $j(\kappa)$ -complete and $2^\lambda < j(\kappa)$, that $\cap j \restriction \nu \in j(\nu)$. Pick $s \in \cap j \restriction \nu$. Since $s \in j(D)$, it follows that $s \in M$, and hence also $s \in V$. Also, $X \in \nu \leftrightarrow s \in j(X)$. Therefore, I can tell in V whether $X_\alpha \in \nu$ by checking whether $s \in j(X_\alpha)$. So $\nu \cap V \in V$, as desired. $\square_{\text{Corollary}}$

Corollary 1.9 *Suppose that 2^κ is a finite cardinal successor of κ , that $V[G]$ admits a gap below κ , and that κ is measurable in $V[G]$. Then every measure on κ in $V[G]$ extends a measure in V .*

Proof: For normal measures on κ , this is a special case of the Local Improvements 1.7 of the Gap Forcing Corollary. But nearly the same argument works for any measure on κ . Suppose ν is a measure on κ in $V[G]$, with embedding $j : V[G] \rightarrow$

$M[j(G)]$. By the Gap Forcing Theorem 1.5 it follows that $M \subseteq V$ and $j'' 2^\kappa \in V$, since $2^\kappa = \kappa^{+(n)}$ for some finite n . It follows that $\langle j(X_\alpha) \mid \alpha < 2^\kappa \rangle \in V$ where $\langle X_\alpha \mid \alpha < 2^\kappa \rangle$ enumerates $P(\kappa)$ in V . Since $X \in \nu \leftrightarrow [\text{id}]_\nu \in j(X)$, I can decide in V whether $X_\alpha \in \nu$. So $\nu \cap V \in V$, and thus ν extends a measure on κ in V . $\square_{\text{Corollary}}$

Let us say that a poset admits a *very low* gap if it admits a gap at or below the least inaccessible cardinal. (This could be modified, without affecting any of the arguments below, to the least Mahlo cardinal, the least weakly compact cardinal, or indeed anything that is strictly below what might become the least measurable cardinal in a forcing extension; e.g. the least weakly compact limit of weakly compact cardinals.)

No Turn-On Corollary 1.10 *If \mathbb{P} admits a very low gap, then it creates no supercompact cardinals. And, if the GCH holds in V , neither does it increase the degree of supercompactness of any cardinal. In particular, it creates no new measurable cardinals.*

Proof: This follows immediately from the Local Improvements 1.7 of the Gap Forcing Corollary, since \mathbb{P} admits a gap below all the cardinals in question. And I don't actually need the full GCH; rather, to show every λ -supercompactness measure in $V[G]$ extends a measure in V , I need only know that $2^{\lambda^{<\kappa}}$ is a finite successor cardinal of λ , and even less than this if \mathbb{P} is distributive. $\square_{\text{Corollary}}$

The Gap Forcing Theorem 1.5 and its corollaries point at the following conjecture, which, if true, would explain and unify its conclusions.

Gap Forcing Conjecture 1.11 *After forcing with a gap below κ , every supercompactness embedding with critical point κ lifts an embedding from the ground model.*

The conjecture asserts that if $j : V[G] \rightarrow M[j(G)]$ is a supercompactness embedding with critical point κ , then $j \restriction V$ is a definable class in V . The conjecture does not assert that $j \restriction V$ is a supercompactness embedding in V ; I have already pointed out, in my remarks preceding the Gap Forcing Theorem, how that can fail. An affirmative answer to the following question would strongly unify the consequences of the Gap Forcing Theorem:

Question 1.12 *After forcing which admits a gap below κ , does every ultrapower embedding by a κ -complete measure on any set lift an embedding from the ground model?*

A even stronger version of this question asks: after forcing which admits a gap below κ , does every embedding with critical point κ lift an embedding from the

ground model? But this is too strong. A counterexample is provided by the situation in which there is a measurable cardinal κ with two distinct normal measures μ and ν . Add a Cohen real x and then add a Cohen subset to ω_2 . This admits a gap at ω_1 , and the measures μ and ν lift and extend uniquely to measures $\bar{\mu}$ and $\bar{\nu}$, respectively, in $V[G]$. The embedding $j : V[G] \rightarrow M[j(G)]$ obtained by iterating the measures $\bar{\mu}$ and $\bar{\nu}$ in the order specified by x cannot be the lift of an embedding definable in V , because from $j \restriction V$ one can recover the real x . Thus, after forcing which admits a gap below κ , it needn't be that every embedding with critical point κ lifts an embedding from the ground model.

§2 Partial Laver Preparations

Laver's crucial contribution in [Lav78] was the idea of what is now called a *Laver function*. For a supercompact cardinal κ , this is a function $\ell : \kappa \rightarrow V_\kappa$ such that for any $\lambda \geq \kappa$ and any $x \in H(\lambda^+)$ there is a λ -supercompact embedding $j : V \rightarrow M$ such that $j(\ell)(\kappa) = x$. The function ℓ can be defined inductively: for a measurable non-supercompact γ , let λ be least such that for some $x \in H(\lambda^+)$ there is no λ -supercompact embedding $j : V \rightarrow M$ with critical point γ such that $j(\ell \restriction \gamma)(\gamma) = x$, select a witness x , and let $\ell(\gamma) = x$. Since this definition is local, in the sense that $\ell(\gamma)$ depends only on choices made concerning $\ell \restriction \gamma$, and not on the ultimate length of the function ℓ to be defined, it actually gives a class function ℓ , called the *universal Laver function*, whose initial segments work as a Laver function simultaneously for every supercompact cardinal (see [KimMag], [Apt96] for elaboration). There are several other simplifying assumptions to be made about a Laver function ℓ , and in this paper I will take them to be part of the definition of what it means to be a Laver function. First of all, I may assume that $\text{dom}(\ell)$ consists entirely of measurable non-supercompact cardinals. Furthermore, I may assume that if $\gamma \in \text{dom}(\ell)$ then γ is closed under ℓ in the sense that $\ell \restriction \gamma \subseteq V_\gamma$. Third, I may assume, as I point out in [Ham94a], that for every λ and every $x \in H(\lambda^+)$ there is a λ -supercompact embedding $j : V \rightarrow M$ such that $j(\ell)(\kappa) = x$ and $\text{dom}(j(\ell)) \cap (\kappa, \lambda] = \emptyset$. Thus, a Laver function has long gaps in its domain.

The *Laver preparation* is the forcing iteration defined by Laver [Lav78] from a Laver function ℓ . It has reverse Easton support, so that direct limits are taken at inaccessible stages, and inverse limits otherwise. The stage γ forcing is exactly $\ell(\gamma)$, provided, as perhaps seems unlikely, but as actually occurs on a stationary set of γ , that this is the \mathbb{P}_γ name of a $<\gamma$ -directed closed poset in $V^{\mathbb{P}_\gamma}$. If $\ell(\gamma)$ is not such

a name, then the stage γ forcing is trivial. Laver proved that performing the Laver preparation makes a supercompact cardinal κ *indestructible*, so that any further $<\kappa$ -directed closed forcing will preserve the supercompactness of κ . The *universal* Laver preparation is simply the Laver preparation obtained by using a universal Laver function, and it has the effect of making every supercompact cardinal indestructible [KimMag], [Apt96].

In what may be the obvious choice of poset, given that I want to make a supercompact cardinal *partially* indestructible, I define that a poset \mathbb{P} is a *partial* Laver preparation of κ iff there is a Laver function ℓ on κ and a set $A \subseteq \text{dom}(\ell)$, the set of *allowed* stages, such that \mathbb{P} is the reverse Easton κ -iteration which at stage γ forces with the poset $\dot{\mathbb{Q}}_\gamma = \ell(\gamma)$, provided, as usual, that $\ell(\gamma)$ is the \mathbb{P}_γ -name of a poset which is $<\gamma$ -directed closed in $V^{\mathbb{P}_\gamma}$, but also, that γ is an allowed stage, i.e. that $\gamma \in A$. It is clear that any partial Laver preparation admits a gap between any two allowed stages.

Partial Laver Preparation Lemma 2.1 *After a partial Laver preparation of a supercompact cardinal κ , it remains supercompact.*

Proof: The usual Laver [Lav78] argument adapts to this circumstance. Suppose $A \subseteq \kappa$ is the set of allowed stages, and assume that $G \subseteq \mathbb{P}$ is V -generic. If A is bounded below κ , then the forcing is small, and so κ remains supercompact. Otherwise, the generic G has size κ . Fix any λ , and $\theta \gg \lambda$, and pick a θ -supercompact embedding $j : V \rightarrow M$ such that $j(\ell)(\kappa)$ is not the name of a poset, and $\text{dom}(j(\ell)) \cap (\kappa, \theta] = \emptyset$. Thus, since there can be no allowed stages from κ to θ , I can factor $j(\mathbb{P})$ as $\mathbb{P} * \mathbb{P}_{\text{tail}}$, where \mathbb{P}_{tail} is $\leq \theta$ -closed in $M[G]$. Thus, it is also $\leq \theta$ -closed in $V[G]$. Let $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ be $V[G]$ -generic, and lift the embedding in $V[G][G_{\text{tail}}]$ to $j : V[G] \rightarrow M[j(G)]$ where $j(G) = G * G_{\text{tail}}$. Now, use $j \restriction \lambda$ as a seed to define a measure μ on $P_\kappa \lambda$ according to the rule $X \in \mu \leftrightarrow j \restriction \lambda \in j(X)$. It is straightforward to check that μ is a normal fine measure on $P_\kappa \lambda$. Furthermore, μ must be in $V[G]$, since it cannot have been added by the closed forcing G_{tail} . Thus, κ remains λ -supercompact in $V[G]$. \square_{Lemma}

No Turn-on Lemma 2.2 *Except possibly for the cardinal γ which is the first nontrivial stage of forcing, a partial Laver preparation adds no supercompact cardinals. If the GCH holds in the ground model, then, except again for γ , neither does it increase the degree of supercompactness of any cardinal. (By adding a Cohen real in front of the preparation, or any other small forcing, these provisos about γ can be removed.)*

Proof: This follows immediately from the Gap Forcing Corollary 1.6 and the Local Improvements 1.7 in the previous section, since the partial Laver preparation admits

a gap just after its first stage. By adding a Cohen real in front, or indeed any small enough forcing, a very low gap is introduced, and the conclusion applies even at γ . \square_{Lemma}

Without the forcing in front of the preparation, it is possible that a partial Laver preparation could increase the degree of supercompactness of its very first stage γ . This would occur, for example, if γ had been previously supercompact, and the current model had been obtained by forcing with reverse Easton support to add subsets to the measurable elements of a certain club of cardinals below γ . It can be arranged that γ would be measurable but not 2^γ -supercompact after this forcing, but then become suddenly supercompact again after the first stage of the subsequent partial preparation.

§3 Jumping High

The concept of a high jump function will prove useful later on in making the tail of an iteration sufficiently closed. A *high jump* function for a (partially) supercompact cardinal κ is a function $h : \kappa \rightarrow \kappa$ such that $j(h)(\kappa) > \lambda$ whenever j is a λ -supercompact embedding on κ .

High Jump Lemma 3.1 *Suppose κ is supercompact but no normal measure on κ concentrates on the supercompact cardinals below κ . Then there is a high jump function for κ . In particular, the least supercompact cardinal has a high jump function.*

Proof: If $\beta < \kappa$ is not supercompact, let $h(\beta) = 2^{\zeta^{<\beta}}$, where ζ is least such that β is not ζ -supercompact. Since κ is supercompact, it follows that $h(\beta) < \kappa$ and so $h : \kappa \rightarrow \kappa$. Now suppose $j : V \rightarrow M$ is a λ -supercompact embedding. Necessarily κ is not supercompact in M , since the induced normal measure does not concentrate on the supercompact cardinals below κ . Thus κ fails in M to be ζ -supercompact for some minimal ζ . If $2^{\zeta^{<\kappa}} \leq \lambda$ in M , then I may code a ζ -supercompactness measure on $P_\kappa \zeta$ in V with a subset of λ . Since $P(\lambda)^M = P(\lambda)^V$, this set, and hence also the measure, must be in M , a contradiction. Therefore, $j(h)(\kappa) = (2^{\zeta^{<\kappa}})^M > \lambda$, as desired. \square_{Lemma}

Another High Jump Lemma 3.2 *If there are fewer than κ many measurable cardinals above the supercompact cardinal κ , then κ has a high-jump function.*

Proof: Suppose there are exactly α many measurable cardinals above κ , and $\alpha < \kappa$. For $\gamma < \kappa$, let $h(\gamma)$ be the $(\alpha + 1)^{\text{th}}$ measurable cardinal above γ . Thus, $h : \kappa \rightarrow \kappa$,

and if $j : V \rightarrow M$ is a λ -supercompact embedding, then $j(h)(\kappa)$ is the $(\alpha + 1)^{\text{th}}$ measurable cardinal above κ in M . Up to λ , however, if a cardinal is measurable in M then it is measurable in V , so the $(\alpha + 1)^{\text{th}}$ measurable cardinal in M is above λ . That is, $j(h)(\kappa) > \lambda$, as desired. \square_{Lemma}

This technique can be pushed much harder. For example, if there are κ many measurable cardinals above κ , one uses the function in which $h(\gamma)$ is the γ^{th} measurable cardinal above γ . If there are κ^+ many, let $h(\gamma)$ be the $\gamma^{+^{\text{th}}}$ measurable above γ . These ideas lead naturally to the ideas of §5. Also, though, in a different sort of generalization, κ need not be *fully* supercompact; if, for example, κ is partially supercompact (but still a limit of measurables), then the function h defined in the proof will be a high-jump function for embeddings up to the degree of supercompactness of κ .

The next theorem shows how the existence of a high jump function is robust.

High Jump Preservation Lemma 3.3 *Suppose that h is a high jump function for κ in V . Then it remains so in any forcing extension $V[G]$ in which every supercompactness measure extends a measure in V . In particular, if $V[G]$ admits a gap below κ and either the GCH holds in V or κ is sufficiently supercompact in $V[G]$, then h is a high-jump function for κ in $V[G]$.*

Proof: Suppose that ν is a λ -supercompactness measure in $V[G]$, with the corresponding embedding $j : V[G] \rightarrow M[j(G)]$, and ν extends a λ -supercompactness measure μ in V . Let $X = \{j(f)(j \restriction \lambda) \mid f \in V\} \prec M$ be the seed hull of $j \restriction \lambda$ via $j \restriction V$. Since X is isomorphic to M_μ , the ultrapower of V by μ , by the map $\varphi : j(f)(j \restriction \lambda) \mapsto [f]_\mu$, it follows that $j_\mu = \pi \circ j$, where π is the collapse of X (See [Ham97b] for elaboration on this seed hull factor method). Consequently, since $\lambda < j_\mu(h)(\kappa)$, it follows that $\lambda < \pi(j(h)(\kappa)) = \pi(j(h)(\kappa)) \leq j(h)(\kappa)$, as desired.

If either the GCH holds in V or κ is sufficiently supercompact in $V[G]$, then the Gap Forcing Corollary 1.6 and the Local Improvements 1.7 yield the necessary hypothesis that every supercompactness measure in $V[G]$ extends a measure in V . For h to work with λ -supercompactness embeddings in $V[G]$, one needs to know either that $2^{\lambda^{<\kappa}}$ is a finite cardinal successor to λ in V (less if the forcing is distributive) or that κ is $2^{\lambda^{<\kappa}}$ -supercompact in $V[G]$. \square_{Lemma}

One might suspect that there is a high jump function for any supercompact cardinal; but the following theorem should temper that tendency.

Almost Huge High Jump Theorem 3.4 *If κ is almost huge, then there is no high jump function for κ .*

Proof: Suppose $j : V \rightarrow M$ is an almost huge embedding, so that $M^{<j(\kappa)} \subseteq M$. It follows that $j(\kappa)$ is regular, and so the set $\{j(h)(\kappa) \mid h \in \kappa^\kappa\}$, which has size 2^κ , is bounded by some $\lambda < j(\kappa)$. Let μ be the supercompactness embedding germinated by the seed $j \restriction \lambda$, and let π be the collapse of the seed hull $X = \{j(f)(j \restriction \lambda) \mid f \in V\} \prec M$. It follows that $j_\mu = \pi \circ j$, and thus, for any function $h \in \kappa^\kappa$ we have $j_\mu(h)(\kappa) = \pi(j(h))(\kappa) = \pi(j(h)(\kappa)) < \pi(\lambda) \leq \lambda$. So there can be no high jump function that works with the measure μ . \square_{Theorem}

Nevertheless, I will show how to add high jump functions for every supercompact cardinal, and moreover, to do this in a way which preserves all supercompact cardinals. By the previous theorem, all almost huge cardinals will of necessity be destroyed. In this argument, I will use the concept of coherent clubs, which were first introduced by Hugh Woodin [W] in his construction to obtain a model of a supercompact cardinal whose weak compactness is easily destroyed. They later returned with a vengeance—huge infinities of them piling up all around—as the central technique, and the central complication, of the Fragile Measurability Theorem [Ham94a]. They appear also in the epilogue of this paper, when I use them to separate the notions of fragility and superdestructibility.

Coherent Club Lemma 3.5 *While preserving all supercompact cardinals, and in fact making them indestructible, one can add by forcing a system of clubs $C_\gamma \subseteq \gamma$ for inaccessible cardinals γ , each disjoint from the supercompact cardinals. Furthermore, the clubs can be made to cohere in the sense that if δ is an inaccessible cluster point of C_γ , then $C_\gamma \cap \delta = C_\delta$. The forcing does not create any supercompact cardinals, and if the GCH holds in the ground model, neither does it increase the degree of supercompactness of any cardinal; in fact, every new supercompactness measure extends an old measure.*

Proof: I will interweave the universal Laver preparation with the forcing to add a system of coherent clubs avoiding the supercompact cardinals. Specifically, let ℓ be a universal Laver function, and let \mathbb{P} be the following reverse Easton class iteration: at inaccessible stages γ , the stage γ forcing has two parts. First is the coherent club forcing \mathbb{Q}_γ , whose conditions are closed bounded subsets $C \subseteq \gamma$, ordered by end-extension, such that C contains no supercompact cardinals of V , and if δ is an inaccessible cluster point of C , then $C \cap \delta = C_\delta$, the club added earlier at stage δ . This forcing has open dense sets as closed as you like up to γ : the set of conditions mentioning a point above $\lambda < \gamma$ is $\leq \lambda$ -directed closed. The second part of the stage γ forcing, \mathbb{R}_γ , is simply the forcing given by the Laver function $\ell(\gamma)$, if this is

the name of $<\gamma$ -directed closed poset in $V^{\mathbb{P}_\gamma * \mathbb{Q}_\gamma}$ (if not, then \mathbb{R}_γ is trivial). Please observe that this iteration admits a very low gap.

Suppose now that G , a proper class, is V -generic for the forcing \mathbb{P} . Let me show that every supercompact cardinal of V is preserved to $V[G]$. Suppose that κ is supercompact in V . I will show that in fact κ becomes indestructible in $V[G]$. So suppose $H \subseteq \mathbb{Q}$ is $V[G]$ -generic for the $<\kappa$ -directed closed forcing \mathbb{Q} . Fix λ , and pick $\theta \gg \lambda, |\mathbb{Q}|$. It suffices to show that κ remains λ -supercompact after forcing with $\mathbb{P}_\theta * \mathbb{Q}$. Factor \mathbb{P}_θ as $\mathbb{P}_\kappa * \mathbb{Q}_\kappa * \mathbb{P}_{\kappa, \theta}$, where \mathbb{Q}_κ is the stage κ coherent club forcing, and $\mathbb{P}_{\kappa, \theta}$ is the rest of the forcing up to stage θ , beginning with \mathbb{R}_κ . Thus, $\mathbb{P}_{\kappa, \theta}$ has a $<\kappa$ -directed closed dense set in $V^{\mathbb{P}_\kappa * \mathbb{Q}_\kappa}$. I may replace $\mathbb{P}_{\kappa, \theta}$ with this dense set and assume that $\mathbb{P}_{\kappa, \theta}$ itself is $<\kappa$ -directed closed. Fix $j : V \rightarrow M$ a θ -supercompact embedding in V such that $j(\ell)(\kappa) = \mathbb{P}_{\kappa, \theta} * \mathbb{Q}$ and $\text{dom}(j(\ell)) \cap (\kappa, \theta] = \emptyset$. In particular, κ is not supercompact in M , since $\text{dom}(\ell)$ contains no supercompact cardinals. Observe that $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{Q}_\kappa * (\mathbb{P}_{\kappa, \theta} * \mathbb{Q}) * \mathbb{P}_{\text{tail}} = P_\theta * \mathbb{Q} * \mathbb{P}_{\text{tail}}$ where \mathbb{P}_{tail} is $<\theta$ -closed in $M^{\mathbb{P}_\theta * \mathbb{Q}}$, and hence also in $V[G][H]$. Let G_{tail} be $V[G][H]$ -generic for \mathbb{P}_{tail} , and then lift to $j : V[G_\kappa] \rightarrow M[j(G_\kappa)]$ where $j(G_\kappa) = G_\kappa * C_\kappa * (G_{\kappa, \theta} * H) * G_{\text{tail}}$. That is, $j(G_\kappa) = G_\theta * H * G_{\text{tail}}$. Now let $\bar{C}_\kappa = C_\kappa \cup \{\kappa\}$. This is a condition in $j(\mathbb{Q}_\kappa)$ since, first, κ is not supercompact in M , second, the reflection $\bar{C}_\kappa \cap \kappa = C_\kappa$ is the generic used at stage κ in $j(G_\kappa)$, and, third, the reflection property holds below κ since C_κ was generic. To use suggestive notation, let $j(C_\kappa)$ be $V[G][H][G_{\text{tail}}]$ -generic for $j(\mathbb{Q}_\kappa)$ below the master condition \bar{C}_κ . This forcing has a dense set which is $\leq \theta$ -directed closed. Now lift the embedding, in $V[G][H][G_{\text{tail}}][j(C_\kappa)]$, to $j : V[G_\kappa][C_\kappa] \rightarrow M[j(G_\kappa)][j(C_\kappa)]$. Observe that $j \restriction G_{\kappa, \theta} \in M[j(G_\kappa)][j(C_\kappa)]$, and $j(\mathbb{P}_{\kappa, \theta})$ is $<j(\kappa)$ -directed closed in that model. Thus, I can find a master condition below $j \restriction G_{\kappa, \theta}$, force below it, and lift to $j : V[G_\kappa][C_\kappa][G_{\kappa, \theta}] \rightarrow M[j(G_\kappa)][j(C_\kappa)][j(G_{\kappa, \theta})]$. That is, $j : V[G_\theta] \rightarrow M[j(G_\theta)]$. Finally, use $j \restriction H$ as a master condition, add a generic $j(H)$, and lift to $j : V[G_\theta][H] \rightarrow M[j(G_\theta)][j(H)]$. This lift is defined in $V[G_\theta][H][G_{\text{tail}}][j(C_\kappa)][j(G_{\kappa, \theta})][j(H)]$. Using $j \restriction \lambda$ as a seed, I germinate a normal fine measure μ on $(P_\kappa \lambda)^{V[G_\theta][H]}$. Since the extra tail forcing and master condition forcing was $<\theta$ -closed, it could not have added μ , and consequently μ lives in $V[G_\theta][H]$, as desired. So every supercompact cardinal in V remains supercompact—becoming in fact indestructible—in $V[G]$.

Since \mathbb{P} admits a very low gap, the No Turn-On Lemma tells us that it cannot create any supercompact cardinals, and, if the GCH holds, neither can it increase the degree of supercompactness of any cardinal; every new supercompactness measure

extends an old measure. Thus, since the clubs which I added are disjoint from the supercompact cardinals of V , they are also disjoint from the supercompact cardinals of $V[G]$. Also, since the clubs are built from initial segments with the coherence property, the clubs themselves also have the coherence property. \square_{Lemma}

Universal High Jump Theorem 3.6 *While preserving all supercompact cardinals and creating no new supercompact cardinals, one can, via forcing with a very low gap, add high jump functions for every supercompact cardinal.*

Proof: If every inaccessible cardinal has a club subset disjoint from the smaller supercompact cardinals, then no normal measure can concentrate on a set of supercompact cardinals. So the corollary follows from the previous lemma and the High Jump Lemma. In fact, after adding the clubs, the proof of the High Jump Lemma produces a single class function $h : \text{ORD} \rightarrow \text{ORD}$ whose restriction $h \upharpoonright \kappa$ to any supercompact cardinal κ yields a high jump function for κ . \square_{Theorem}

High Jump Theorem 3.7 *While preserving all supercompact cardinals and creating no new supercompact cardinals, via forcing with a very low gap, one can add high jump functions for every supercompact cardinal up to and including κ , without collapsing cardinals above κ . If the GCH holds, then it can also be arranged to add the functions without collapsing cardinals or cofinalities at all.*

Proof: For the first part, just perform the coherent club forcing up to and including the stage κ coherent club forcing. This forcing has size κ , and hence preserves all cardinals above κ . For the second part, when the GCH holds, one can perform a modified coherent club forcing in which the stage γ forcing is allowed only when, in addition, it preserves all cardinals, cofinalities, and the GCH. I will make a similar argument—giving all the details—in Theorem 4.3. \square_{Theorem}

§4 Exact Preservation Theorems

Let me now set off into the unknown continent between the extremes of indestructibility and superdestructibility. My initial explorations will reveal the rich structure to be found there: exact preservation theorems. In these theorems, borrowing from both indestructibility and superdestructibility, I will precisely control the class of $<\kappa$ -directed closed posets which preserve the supercompactness of κ , and obtain models where *exactly* a certain class of posets preserves the supercompactness of a given supercompact cardinal κ . I will lean on Laver's methods to show that a

poset preserves supercompactness, and on the Gap Forcing Theorem 1.5 and its corollaries to show that a poset destroys supercompactness.

Let me begin with two warm-up theorems, in which I show that the notions of superdestructibility at κ and at κ^+ are orthogonal. Recall from [Ham97b] that a supercompact cardinal κ is *superdestructible at θ* when any $<\kappa$ -closed forcing which adds a subset to θ destroys the θ -supercompactness of κ .

Exact Preservation Theorem 4.1 *Assume that κ is supercompact in V . Then there is a forcing extension $V[G]$ in which κ remains supercompact and becomes superdestructible at κ^+ but not at κ .*

Proof: I will obtain a model where κ is supercompact and any $<\kappa$ -closed poset which adds a subset to κ^+ destroys the κ^+ -supercompactness of κ , but where the measurability of κ is preserved by any $<\kappa$ -directed closed poset which preserves κ^+ and $2^\kappa = \kappa^+$.

To begin I may assume, by forcing if necessary, that $2^\kappa = \kappa^+$ in V . After this, let \mathbb{P} be the partial Laver preparation of κ in which the stage γ forcing is allowed only when it destroys the measurability of γ , but preserves γ^+ and $2^\gamma = \gamma^+$, if indeed this held in V . Suppose that $G \subseteq \mathbb{P}$ is V -generic. By the Partial Laver Preparation Lemma, I know that κ remains supercompact in $V[G]$.

Let me now prove that κ is superdestructible at κ^+ in $V[G]$. Suppose that $H \subseteq \mathbb{Q}$ is $V[G]$ -generic, where \mathbb{Q} is a $<\kappa$ -closed forcing notion which adds a new subset $B \subseteq \kappa^+$, but that κ remains κ^+ -supercompact in $V[G][H]$. (Here I mean κ^+ to denote $(\kappa^+)^V = (\kappa^+)^{V[G]}$.) Thus, there is a κ^+ -supercompact embedding $j : V[G][H] \rightarrow M[j(G)][j(H)]$. By the Gap Forcing Theorem 1.5, I know $P(\kappa^+)^M = P(\kappa^+)^V$, and thus $2^\kappa = \kappa^+$ in M . If κ^+ is collapsed by H , then it must also be collapsed by $j(G)$, which is impossible since the only stage which could do this is the stage κ forcing, and that stage is not allowed if it collapses κ^+ . Thus, H must not collapse κ^+ . If the stage κ forcing is not allowed, then $j(G) = G * G_{\text{tail}}$, where G_{tail} is $\leq \kappa^+$ -closed, and consequently $B \in M[G] \subseteq V[G]$, a contradiction. If the stage κ forcing is allowed, then $j(G) = G * g * G_{\text{tail}}$, where g is the stage κ forcing and G_{tail} is again $\leq \kappa^+$ -closed. Since κ was allowed I know that κ is not measurable in $M[G][g]$. But $P(\kappa^+)^{M[G][g]} = P(\kappa^+)^{M[j(G)][j(H)]} = P(\kappa^+)^{V[G][H]}$, and so by coding a measure on κ from $V[G][H]$ into a subset of κ^+ I conclude that it lies in $M[G][g]$, and so κ is measurable there after all, a contradiction.

Finally, I will show that in $V[G]$ the measurability of κ is preserved by any $<\kappa$ -directed closed forcing which preserves κ^+ and $2^\kappa = \kappa^+$. Suppose $H \subseteq \mathbb{Q}$ is generic

for such a poset, but that κ is not measurable in $V[G][H]$. Fix a large θ and a θ -supercompact embedding $j : V \rightarrow M$ such that $j(\ell)(\kappa) = \dot{\mathbb{Q}}$ and $\text{dom}(j(\ell)) \cap (\kappa, \theta] = \emptyset$. Since κ is not measurable in $V[G][H]$, this is also true in $M[G][H]$, and, similarly, H preserves κ^+ and $2^\kappa = \kappa^+$ over $M[G]$. Thus, κ is an allowed stage in the $j(\mathbb{P})$ forcing, and so $j(\mathbb{P})$ factors as $\mathbb{P} * \mathbb{Q} * \mathbb{P}_{\text{tail}}$ where \mathbb{P}_{tail} is $\leq \theta$ -closed. Force over the tail and lift the embedding to $j : V[G] \rightarrow M[j(G)]$ where $j(G) = G * H * G_{\text{tail}}$ in $V[G][H][G_{\text{tail}}]$. Observe that $j \restriction H \in M[j(G)]$, and so by the directed closure of $j(\mathbb{Q})$, there is a master condition $p \in j(\mathbb{Q})$ which is below every element of $j \restriction H$. Let $j(H)$ be generic below p , and lift the embedding to $j : V[G][H] \rightarrow M[j(G)][j(H)]$ in $V[G][H][G_{\text{tail}}][j(H)]$. Using κ as a seed, I obtain a normal measure μ on $P(\kappa)^{V[G][H]}$. By closure considerations I know μ must be in $V[G][H]$, and so κ is measurable there. \square_{Theorem}

Next, for the second warm-up theorem, I will prove the opposite combination. Define that a supercompact cardinal κ is *indestructible above θ* iff any $< \kappa$ -directed closed forcing which adds no subsets to θ preserves the supercompactness of κ .

Exact Preservation Theorem 4.2 *Assume κ is supercompact in V . Then there is a forcing extension $V[G]$ in which κ remains supercompact and becomes superdestructible at κ but not at κ^+ . Indeed, κ can be made simultaneously superdestructible at κ and indestructible above κ .*

Proof: Here I will perform the partial Laver preparation of κ in which the stage γ forcing is allowed provided that it adds no new subsets to γ . Let me emphasize here for clarity that it is part of the definition of a partial Laver preparation that, in addition, the forcing \mathbb{Q}_γ which the Laver function hands to us at stage γ must be $< \gamma$ -directed closed in $V^{\mathbb{P}_\gamma}$. Suppose that $G \subseteq \mathbb{P}$ is V -generic for this forcing. By the Partial Laver Preparation Lemma, it follows that κ remains supercompact in $V[G]$.

If κ is not superdestructible at κ in $V[G]$, then there must be some $< \kappa$ -closed forcing \mathbb{Q} , and a $V[G]$ -generic $H \subseteq \mathbb{Q}$, adding a new subset $B \subseteq \kappa$, such that κ remains measurable in $V[G][H]$, with the corresponding embedding $j : V[G][H] \rightarrow M[j(G)][j(H)]$. By closure considerations, $B \in M[G]$, since the stage κ forcing cannot have added a subset to κ . But the Gap Forcing Theorem 1.5 tells us that $M[G] \subseteq V[G]$, and so $B \in V[G]$, contradicting the fact that it was newly added by H . So κ is superdestructible in $V[G]$, as desired.

To show that κ becomes indestructible above κ in $V[G]$, I will employ what I will later refer to as the ‘usual’ lifting argument: suppose $H \subseteq \mathbb{Q}$ is $V[G]$ -generic for

$<\kappa$ -directed closed forcing \mathbb{Q} which adds no new subsets to κ . Fix any λ , and pick $\theta \gg \lambda, |\mathbb{Q}|$. Select a θ -supercompact embedding $j : V \rightarrow M$ such that $j(\ell)(\kappa) = \dot{\mathbb{Q}}$ and $\text{dom}(j(\ell)) \cap (\kappa, \theta] = \emptyset$. The stage κ is allowed since \mathbb{Q} adds no subsets to κ . Thus, $j(\mathbb{P}) = \mathbb{P} * \mathbb{Q} * \mathbb{P}_{\text{tail}}$, where \mathbb{P}_{tail} is $\leq \theta$ -closed. Force to add a generic $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$, and lift the embedding to $j : V[G] \rightarrow M[j(G)]$, where $j(G) = G * H * G_{\text{tail}}$. The lift is defined in $V[G][H][G_{\text{tail}}]$. Now $j(\mathbb{Q})$ is $<j(\kappa)$ -directed closed, and $j''H \in M[j(G)]$, so I can find a master condition below $j''H$ in $j(\mathbb{Q})$, and force below it to add a generic $j(H) \subseteq j(\mathbb{Q})$. This gives a lift embedding $j : V[G][H] \rightarrow M[j(G)][j(H)]$, defined in $V[G][H][G_{\text{tail}}][j(H)]$. Use $j''\lambda$ as a seed to germinate a normal fine measure μ on $P_\kappa \lambda^{V[G][H]}$ as follows: $X \in \mu \leftrightarrow j''\lambda \in j(X)$. Since the forcing $G_{\text{tail}} * j(H)$ was $\leq \theta$ -closed, it could not have added μ , and consequently $\mu \in V[G][H]$. Thus, κ is λ -supercompact there, as desired. \square_{Theorem}

I can in addition exhibit models of the third and fourth possibilities: after small forcing, the main result of [Ham97b] shows that a supercompact cardinal is superdestructible at both κ and κ^+ ; and if κ is indestructible, then κ is superdestructible at neither κ nor κ^+ . Let me now continue with additional Exact Preservation Theorems.

Exact Preservation Theorem 4.3 *Assume κ is supercompact and the SCH holds in V . Then there is a forcing extension $V[G]$, obtained without collapsing cardinals or cofinalities, in which κ remains supercompact and indeed over $V[G]$ the supercompactness of κ is preserved by exactly those $<\kappa$ -directed closed posets which collapse neither cardinals nor cofinalities.*

Proof: Let \mathbb{P} be the partial Laver preparation of κ in which stage γ is allowed provided that it collapses neither cardinals nor cofinalities, and suppose that $G \subseteq \mathbb{P}$ is V -generic. By the Partial Laver Preparation Lemma, κ remains supercompact in $V[G]$.

Let me argue that the supercompactness of κ is preserved by any $<\kappa$ -directed closed forcing which collapses neither cardinals nor cofinalities. Suppose \mathbb{Q} is such forcing, and $H \subseteq \mathbb{Q}$ is $V[G]$ -generic. Fix λ , pick $\theta \geq 2^\lambda, |\mathbb{Q}|$, and select as usual $j : V \rightarrow M$ a θ -supercompact embedding such that $j(\ell)(\kappa) = \dot{\mathbb{Q}}$ and $\text{dom}(j(\ell)) \cap (\kappa, \theta] = \emptyset$. Since H preserves cardinals and cofinalities over $V[G]$, it follows by the largeness of θ that this is also true over $M[G]$, and so the stage κ forcing is allowed. By the usual lifting argument, lift to $j : V[G][H] \rightarrow M[j(G)][j(H)]$ and conclude that κ remains λ -supercompact in $V[G][H]$.

Conversely, I will also argue that the supercompactness of κ is destroyed by any

$<\kappa$ -closed forcing \mathbb{Q} which collapses cardinals or cofinalities. Necessarily, \mathbb{Q} must collapse the cofinality of some regular cardinal. Suppose $H \subseteq \mathbb{Q}$ is $V[G]$ -generic and, for some regular cardinal λ in $V[G]$, I have $\text{cof}(\lambda) = \delta < \lambda$ in $V[G][H]$, but that κ remains δ -supercompact in $V[G][H]$. Let $j : V[G][H] \rightarrow M[j(G)][j(H)]$ be the witness embedding. By the closure of the embedding it follows that $\text{cof}(\lambda)^{M[j(G)][j(H)]} = \delta$. By the Gap Forcing Theorem 1.5 I know that M and V have the same δ -sequences of ordinals, and consequently, $\text{cof}(\lambda)^M > \delta$. But $j(G)$ cannot have collapsed any cardinals or cofinalities over M , since, as I will prove in the next paragraph, G did not over V , and $j(H)$ is $<j(\kappa)$ -closed, so it cannot have added the δ -sequence either, a contradiction.

I must now prove that neither cardinals nor cofinalities were collapsed between V and $V[G]$. Here I will use the SCH assumption, but it is not onerous. The SCH follows of course from the GCH, which one can easily force while preserving supercompactness. Also, though, Solovay [Sol74] proved that the SCH holds automatically above any supercompact cardinal κ , and by reflection it must hold unboundedly often below κ . The content of the hypothesis is merely that the SCH holds at the remainder of the singular cardinals below κ . Certainly if κ is not the least supercompact cardinal, then I could have omitted the SCH assumption entirely, by starting the partial Laver preparation beyond the first supercompact cardinal, so that the SCH holds when I need it. But let me begin the argument at hand. It suffices to show that all regular cardinals below κ are preserved. Suppose towards a contradiction that λ is regular in V , but that $\text{cof}(\lambda) = \gamma < \lambda$ in $V[G]$. The cardinal γ must be regular. Factor \mathbb{P} as $\mathbb{P}_\gamma * \mathbb{Q}_\gamma * \mathbb{P}_{\text{tail}}$, where \mathbb{Q}_γ is nontrivial only if γ is an allowed stage. The forcing \mathbb{P}_{tail} is $\leq \gamma$ -closed, and so it could not collapse the cofinality of λ to γ . Similarly, the forcing \mathbb{Q}_γ would not be allowed if it collapsed the cofinality of λ . So I need only check that \mathbb{P}_γ does not collapse the cofinality of λ . Let β be the supremum of the allowed stages before γ ; I really need to check only that \mathbb{P}_β does not collapse the cofinality of λ . By stripping off the successor stages one by one, since these cannot collapse cofinalities, I may assume that β is a limit of allowed stages. Thus, in particular, β is a strong limit cardinal. Since the next stage in a partial Laver preparation does not occur until beyond the size of the previous stage forcing, it follows that $|\mathbb{P}_\alpha| < \beta$ for all $\alpha < \beta$. Now, there are two possibilities. If, on the one hand, β is regular then, being a regular limit of inaccessibles, it follows that β is itself inaccessible. Thus, $|\mathbb{P}_\beta| = \beta$ since I took a direct limit at stage β , and so \mathbb{P}_β , having size β , is β^+ -c.c., and consequently unable to collapse the cofinality of λ . If, on the other hand, β is singular, then $\beta < \gamma$

and also, since I took an inverse limit at stage β , $|\mathbb{P}_\beta| \leq \beta^{\text{cof}(\beta)} = \beta^+$, by the SCH. Thus, \mathbb{P}_β is β^{++} -c.c., and this is good enough since $\beta < \gamma < \lambda \rightarrow \beta^{++} \leq \lambda$, so the cofinality of λ could not have been collapsed by \mathbb{P}_β . In any case, therefore, I obtain a contradiction. So neither cardinals nor cofinalities are collapsed between V and $V[G]$. And this completes the proof. \square_{Theorem}

Exact Preservation Theorem 4.4 *Assume that κ is supercompact in V . Then there is a forcing extension $V[G]$ in which κ remains supercompact, the GCH holds, and over which the supercompactness of κ is preserved by exactly those $<\kappa$ -directed closed posets which preserve the GCH.*

Proof: For this proof, I will not actually need to use the Gap Forcing Theorem 1.5. I may assume, by forcing if necessary, that the GCH holds in V . Suppose that G is V -generic for the partial Laver preparation of κ in which stage γ is allowed provided that it preserves the GCH (cardinals may be collapsed). By the Partial Laver Preparation Lemma, κ remains supercompact in $V[G]$.

First, I will argue that the GCH still holds in $V[G]$. Certainly it still holds at κ and above, since \mathbb{P} has size κ . Suppose that the GCH holds up to $\gamma < \kappa$. I can factor \mathbb{P} as $\mathbb{P}_\gamma * \mathbb{Q}_\gamma * \mathbb{P}_{\text{tail}}$, where \mathbb{Q}_γ is trivial unless γ is allowed. The tail forcing \mathbb{P}_{tail} adds no subsets to γ , and so cannot affect the GCH at γ . The stage γ forcing \mathbb{Q}_γ is only allowed provided that it preserves the GCH. So consider \mathbb{P}_γ . Let β be the supremum of the allowed stages below γ . It suffices to show that the \mathbb{P}_β preserves the GCH at γ . Without loss of generality, by stripping off the successor stages one by one, which cannot affect the GCH, I may assume that β is a limit of allowed stages. Now, if $\beta < \gamma$, then \mathbb{P}_β , being the limit of smaller posets, has size $\beta^+ \leq \gamma$, and so it cannot destroy the GCH at γ . Otherwise, assume $\beta = \gamma$, so γ is a limit of allowed stages, and therefore a strong limit cardinal. If γ is regular, then it must be inaccessible, and so \mathbb{P}_γ , using the direct limit, has size γ , and consequently cannot destroy the GCH at γ . So assume γ is singular. By Silver's theorem [Sil74], it suffices to consider the case that $\text{cof}(\gamma) = \omega$, since otherwise the GCH, holding below γ , automatically holds at γ . But in this case, $2^\gamma = \gamma^\omega$, and since the entire forcing \mathbb{P} is countably closed, it cannot affect γ^ω . Thus, it cannot destroy the GCH at γ . So $V[G] \models \text{GCH}$.

Let me now argue that κ is indestructible by any $<\kappa$ -directed closed forcing which preserves the GCH. Suppose \mathbb{Q} is such forcing, and that $H \subseteq \mathbb{Q}$ is $V[G]$ -generic. Fix λ and pick θ much larger than both λ and $|\mathbb{Q}|$, and a θ -supercompact embedding $j : V \rightarrow M$ such that $j(\ell)(\kappa) = \dot{\mathbb{Q}}$ and $\text{dom}(j(\ell)) \cap (\kappa, \theta] = \emptyset$. Since H

preserves the GCH over $V[G]$, it follows that H also preserves the GCH over $M[G]$, since $M[G]$ and $V[G]$ agree up to θ , which is much larger than H . The stage κ forcing, therefore, is allowed in $j(\mathbb{P})$, and I may continue the usual argument to lift j to $j : V[G][H] \rightarrow M[j(G)][j(H)]$, and then use $j \restriction \lambda$ as a seed to conclude that κ is still λ -supercompact in $V[G][H]$.

Finally, it is easy to see that the supercompactness of κ is destroyed by any $<\kappa$ -closed forcing which does not preserve the GCH, and this is simply because if the GCH holds up to a supercompact cardinal κ , then it must hold everywhere. \square_{Theorem}

Similar arguments establish the next two theorems, whose proofs I omit, except to say that in the first, one uses a partial Laver preparation in which γ is allowed when \mathbb{Q}_γ does not collapse γ^+ , and in the second, γ is allowed when γ is closed under some fixed high jump function $h : \kappa \rightarrow \kappa$ and \mathbb{Q}_γ does collapse γ^+ .

Exact Preservation Theorem 4.5 *Assume that κ is supercompact in V . Then there is a forcing extension $V[G]$ in which κ remains supercompact and over which the supercompactness of κ is preserved by exactly those $<\kappa$ -directed closed posets which do not collapse κ^+ . Indeed, in $V[G]$ a $<\kappa$ -directed closed poset, if it preserves κ^+ , will preserve the supercompactness of κ ; if it collapses κ^+ , it will destroy the measurability of κ .*

Exact Preservation Theorem 4.6 *Assume that κ is supercompact in V . Then there is a forcing extension $V[G]$ in which κ remains supercompact and over which the supercompactness of κ is preserved by exactly those $<\kappa$ -directed closed posets which collapse κ^+ .*

There is no end to these kinds of theorems. In particular, it is an easy matter to change the κ^+ in the previous two theorems to κ^{++} or κ^{+++} and so on: I can get a model where the supercompactness of κ is preserved by exactly those $<\kappa$ -directed closed posets which collapse κ^{++++} to κ^{++} , to name but one way of modifying the theorem. Really it is only due to the Gap Forcing Theorem 1.5 that we have some knowledge about the supercompact embeddings which live in a gap forcing extension of V . In the next section I will introduce a new topic in order to prove, later, even more powerful exact preservation theorems than these.

Remark on Closure 4.7 I have made some mention of forcing notions which are, variously, $<\kappa$ -directed closed, $<\kappa$ -closed, and $<\kappa$ -strategically closed. But how much closure do I need for the arguments? The answer is that I need the forcing \mathbb{Q} to be $<\kappa$ -directed closed when I want to argue that \mathbb{Q} preserves supercompactness; directed closure is used to find a master condition for $j(\mathbb{Q})$. I only need \mathbb{Q} , however, to be $<\kappa$ -closed, or, even weaker, $<\kappa$ -strategically closed, when I want to argue that \mathbb{Q} destroys supercompactness, since this is all that the Gap Forcing Theorem 1.5 requires. I have stated all the Exact Preservation Theorems only for $<\kappa$ -directed closed forcing, for simplicity, but the proofs go through without modification for strategically closed posets on the destruction side, provided one allows strategically closed forcing in the partial Laver preparation itself.

§5 Representability

In order to improve the Exact Preservation Theorems, I will now generalize the beautiful fact of folklore—happily discovered I am sure by many young set theorists, perhaps like myself, while sipping coffee in cafes—that for every ordinal $\alpha < \kappa^+$ there is a function $f : \kappa \rightarrow \kappa$ which represents α with respect to every normal measure on κ . Specifically, $j(f)(\kappa) = \alpha$ for any such ultrapower embedding.

More generally, now, supposing first that $a \in H(\theta^+)$, I will say that a is *represented* by the function $f : \kappa \rightarrow V_\kappa$ with respect to θ -supercompact embeddings exactly when

$$j(f)(\kappa) = a$$

for any such embedding. Generalizing still further, to allow for the possibility that a is much larger than θ , or that a is a proper class, I officially require only that

$$j(f)(\kappa) \cap H(\theta^+) = a \cap H(\theta^+)$$

for all the θ -supercompact embeddings j . This agrees with the first definition when $a \in H(\theta^+)$. Thus I require that $j(f)(\kappa)$ agree with a as well as the embeddings can be expected to make it so. I am referring here only to embeddings with critical point κ . Define that a is *representable* when there is a function $f : \kappa \rightarrow V_\kappa$ which represents a with respect to θ -supercompact embeddings for every $\theta \geq \kappa$. The set a is *eventually* representable if there is a function which represents a with respect to all θ -supercompact embeddings for sufficiently large θ , and a is *frequently* representable if there is a function which represents a with respect to all θ -supercompact embeddings, for arbitrarily large θ .

Folklore Fact 5.1 *Every ordinal below κ^+ is representable.*

Proof: This is the happy fact I referred to above. Certainly any ordinal below κ is representable; one simply uses a constant function. Now suppose $\kappa \leq \alpha < \kappa^+$. Therefore, $\alpha = \text{ot}\langle \kappa, \triangleleft \rangle$ for some relation \triangleleft on κ . Define $f(\gamma) = \text{ot}\langle \gamma, \triangleleft \upharpoonright \gamma \rangle$. Thus, for any $j : V \rightarrow M$ with critical point κ , regardless of where j is defined, it follows that $j(f)(\kappa) = \text{ot}\langle \kappa, j(\triangleleft) \upharpoonright \kappa \rangle = \text{ot}\langle \kappa, \triangleleft \rangle = \alpha$, as desired. \square_{Fact}

But this fact is only the start. What I aim to show now is that there are many more representable ordinals; indeed, the next two closure theorems show that the set of representable sets forms a small set-theoretic universe.

Ordinal Representation Theorem 5.2 *The set of (eventually) representable ordinals is closed under ordinal and cardinal arithmetic. Specifically, if α and β are (eventually) representable, then so are the ordinals $\alpha + \beta$, $\alpha\beta$, α^β , and the cardinals $|\alpha|$, \aleph_α , \beth_α , α^+ , and $|\alpha|^{|\beta|}$. Furthermore, the set of (eventually) representable ordinals is closed under $\leq \kappa$ -suprema.*

Proof: Suppose that α and β are represented by the functions f_α and f_β . It is easy to see that $\alpha + \beta$, etc. are represented by the following functions:

$$f_{\alpha+\beta}(\gamma) = f_\alpha(\gamma) + f_\beta(\gamma) \quad f_{\alpha\beta}(\gamma) = f_\alpha(\gamma) \cdot f_\beta(\gamma) \quad f_{\alpha^\beta}(\gamma) = f_\alpha(\gamma)^{f_\beta(\gamma)}$$

This follows by the absoluteness of ordinal arithmetic between M and V . For example, if j is θ -supercompact and $\alpha, \beta \leq \theta$, then $j(f_{\alpha+\beta})(\kappa) = j(f_\alpha)(\kappa) + j(f_\beta)(\kappa) = \alpha + \beta$, as desired. The cardinals $|\alpha|$, etc. are represented by the functions:

$$\begin{aligned} f_{|\alpha|}(\gamma) &= |f_\alpha(\gamma)| & f_{\aleph_\alpha}(\gamma) &= \aleph_{f_\alpha(\gamma)} & f_{\beth_\alpha}(\gamma) &= \beth_{f_\alpha(\gamma)} \\ f_{\alpha^+}(\gamma) &= f_\alpha(\gamma)^+ & f_{|\alpha|^{|\beta|}}(\gamma) &= |f_\alpha(\gamma)|^{f_\beta(\gamma)} \end{aligned}$$

Again, I need to appeal to absoluteness for these notions between M and V . For illustration, consider \aleph_α . Suppose that $j : V \rightarrow M$ is a θ -supercompact embedding. The argument is a little easier in the case that θ is very large, for then $j(f_\alpha)(\kappa) = \alpha$ and consequently $j(f_{\aleph_\alpha})(\kappa) = \aleph_\alpha^M = \aleph_\alpha$, as required. If θ is less than \aleph_α , then I only need to show that $j(f_{\aleph_\alpha})(\kappa)$ is at least θ^+ . In the case that θ is between α and \aleph_α then $j(f_\alpha)(\kappa) = \alpha$ and consequently $j(f_{\aleph_\alpha})(\kappa) = \aleph_\alpha^M$, which is at least θ^+ . If θ is less than α , then $j(f_\alpha)(\kappa)$ is at least θ^+ , so $j(f_{\aleph_\alpha})(\kappa) = \aleph_{j(f_\alpha)(\kappa)}^M$, which is at least θ^+ . So in any case \aleph_α and $j(f_{\aleph_\alpha})(\kappa)$ agree up to θ^+ , as is required. The other cases are similar.

Finally, let me show that the set of (eventually) representable ordinals is closed under $\leq \kappa$ -sups. Suppose that $\lambda = \sup_{\alpha < \kappa} \lambda_\alpha$, where λ_α is represented by f_{λ_α} . Let $f_\lambda(\gamma) = \sup_{\alpha < \gamma} f_{\lambda_\alpha}(\gamma)$, and observe that $j(f_\lambda)(\kappa) = \sup_{\alpha < \kappa} j(f_{\lambda_\alpha})(\kappa)$. This agrees with λ up to θ^+ since $j(f_{\lambda_\alpha})(\kappa)$ agrees with λ_α up to θ^+ . \square_{Theorem}

Representation Theorem 5.3 *Every element of $H(\kappa^+)$ is representable. The set of (eventually) representable sets is closed under elementary set operations. Specifically, if a and b are (eventually) representable sets, then so are $\{a, b\}$, $a \setminus b$, and $P(a)$; as are the set $\{x \in a \mid \phi(x, b)\}$ and the ordinal $\mu\alpha[\phi(H(\alpha^+), \alpha, a)]$, when ϕ is any Δ_0 formula. If a is eventually representable, then so is $\cup a$. If α is an (eventually) representable ordinal, then V_α is (eventually) representable. Also, the set of (eventually) representable sets is closed under κ -sequences.*

Proof: Suppose that $a \in H(\kappa^+)$. Thus, a can be coded by some set $A \subseteq \kappa$. Let $f_a(\gamma)$ be the set in $H(\gamma^+)$ which is coded in the same way by $A \cap \gamma$, so that $j(f)(\kappa)$ is the set coded by $j(A) \cap \kappa = A$. That is, $j(f)(\kappa) = a$, as required.

The closure claims are similar to the previous theorem. One simply uses the obvious function in each case, and then appeals to absoluteness between M and V . Let me illustrate with $P(a)$. Suppose that a is represented by the function f_a . Define $f_{P(a)}(\gamma) = P(f_a(\gamma))$, and suppose that $j : V \rightarrow M$ is a θ -supercompact embedding. Since $j(f_a)(\kappa)$ agrees with a on $H(\theta^+)$, and M is closed under θ -sequences, it follows that $P(j(f_a)(\kappa))^M$ agrees with $P(a)$ on $H(\theta^+)$, as is required. The other cases are similar. To show the closure under κ -sequences, suppose that a_α is represented by f_α for $\alpha < \kappa$. It is easy to see that the function $f(\gamma) = \langle f_\alpha(\gamma) \mid \alpha < \gamma \rangle$ represents the sequence $\langle a_\alpha \mid \alpha < \kappa \rangle$. \square_{Theorem}

It seems possible that a could be representable but not $\cup a$, because for some θ perhaps a and $j(f)(\kappa)$ disagree about a set which is not in $H(\theta^+)$ but which has elements in $H(\theta^+)$. This is why I only make the claim of eventual representability for $\cup a$ in the theorem.

Going-Up Lemma 5.4 *If f represents a with respect to all λ -supercompact embeddings, and $a \in H(\lambda^+)$, then f represents a with respect to all θ -supercompact embeddings for any $\theta > \lambda$.*

Proof: Suppose that $j : V \rightarrow M$ is a θ -supercompact embedding, for $\theta > \lambda$. Let μ be the λ -supercompactness measure germinated by the seed $j \restriction \lambda$. That is, $X \in \mu \leftrightarrow j \restriction \lambda \in j(X)$. If $j_\mu : V \rightarrow M_\mu$ is the ultrapower by μ , it follows that $a \in M_\mu$, and also that $j_\mu = \pi \circ j$, where π is the collapse of the seed hull $X = \{j(g)(j \restriction \lambda) \mid g \in V\} \prec M$ (consult [Ham97b] for elaboration on this seed hull

factor method), as illustrated in the following diagram.

$$\begin{array}{ccc}
 V & & \\
 \downarrow j_\mu & \searrow j & \\
 M_\mu & \xrightarrow{\pi^{-1}} & M
 \end{array}$$

Since $\alpha = \text{ot}(j \restriction \lambda \upharpoonright j(\alpha))$ for all $\alpha \leq \lambda$ it follows that $\lambda \subseteq X$ and $\lambda \in X$. Thus, $\pi(a) = a$ and $\pi(\kappa) = \kappa$, and consequently, since f represents a with respect to j_μ , it follows that $a = j_\mu(f)(\kappa) = \pi(j(f))(\kappa) = \pi(j(f)(\kappa))$. Thus $j(f)(\kappa) = a$ as desired. \square_{Lemma}

Corollary 5.5 *A set is frequently representable iff it is eventually representable.*

Proof: Immediate from the Going-Up Lemma. $\square_{\text{Corollary}}$

Going-Down Lemma 5.6 *If f represents a with respect to a θ -supercompact embedding, then for every $\lambda < \theta$ there is a λ -supercompact embedding with respect to which f represents a .*

Proof: Suppose f represents a with respect to $j : V \rightarrow M$, a θ -supercompact embedding. Let μ be the measure on $P_\kappa \lambda$ germinated by $j \restriction \lambda$ via j , and let $j_\mu : V \rightarrow M_\mu$ be the corresponding ultrapower embedding. If $x \in H(\lambda^+)$, it follows as in the previous argument that $\pi(x) = x$, and consequently

$$\begin{aligned}
 x \in a &\leftrightarrow x \in j(f)(\kappa) \\
 &\leftrightarrow \pi(x) \in \pi(j(f)(\kappa)) \\
 &\leftrightarrow x \in j_\mu(f)(\kappa).
 \end{aligned}$$

So f represents a with respect to j_μ , a λ -supercompact embedding. \square_{Lemma}

Enduring Representability Theorem 5.7 *Suppose that a class A is represented by the function $f : \kappa \rightarrow V_\kappa$ in V , and that $V[G]$ is a forcing extension which admits a gap below κ . If either (1) κ is supercompact in $V[G]$, or (2) A is a set and κ is $2^{|\text{TC}(A)|}$ -supercompact in $V[G]$, or (3) the GCH holds in V , then f continues to represent A in $V[G]$.*

Proof: Suppose that $j : V[G] \rightarrow M[j(G)]$ is a θ -supercompact embedding in $V[G]$. I need to show, under the various hypotheses, that $j(f)(\kappa)$ agrees with A on $H(\theta^+)$. By the Going-Up Lemma, if A is a set I may assume that $\theta \leq |\text{TC}(A)|$. The

argument is complicated somewhat by the possibility that θ^{+V} may be collapsed by G . Suppose that $x \in H(\theta^{+V[G]})$. I aim to show that $x \in A \leftrightarrow x \in j(f)(\kappa)$. Since $j(f)(\kappa) \in M \subseteq V$, it suffices to consider only the case when $x \in V$. It follows that there is some λ such that $\theta \leq \lambda < \theta^{+V[G]}$ and $x \in H(\lambda^+)^V$. Since $|\lambda| = \theta$ in $V[G]$, I may view j as a λ -supercompact embedding. Let μ be the measure germinated by $j \restriction \lambda$ via $j \restriction V$. By the Local Improvements 1.7 of the Gap Forcing Corollary 1.6, under any of the hypotheses in 1, 2, or 3, this measure, and therefore also the corresponding ultrapower map $j_\mu : V \rightarrow M_\mu$, is in V . Consequently, since f represents A in V , it follows that $j_\mu(f)(\kappa)$ agrees with A on $H(\lambda^+)^V$. Since μ is germinated from $j \restriction \lambda$ via j , it follows that $j_\mu = \pi \circ j$ where π is the collapse of the seed hull $X = \{j(g)(j \restriction \lambda) \mid g \in V\} \prec M$. Since as in the Going-Up Lemma $\lambda \subseteq X$ and $\lambda \in X$, and also $H(\lambda^+)^{M_\mu} = H(\lambda^+)^V$, it follows that $\pi(x) = x$ and $\pi(\kappa) = \kappa$. Now simply compute:

$$\begin{aligned}
 x \in A &\leftrightarrow x \in j_\mu(f)(\kappa) \\
 &\leftrightarrow \pi(x) \in \pi(j_\mu(f)(\kappa)) \\
 &\leftrightarrow x \in \pi(j_\mu(f))(\pi(\kappa)) \\
 &\leftrightarrow x \in j(f)(\kappa).
 \end{aligned}$$

Thus A and $j(f)(\kappa)$ agree on x , as is required. \square_{Theorem}

While the Representation Theorems show that the class of representable sets forms a small set-theoretic universe, we must keep in mind that a single function $f : \kappa \rightarrow V_\kappa$ represents at most one set, and so the number of representable sets is at most 2^κ . Nevertheless, I can make any set representable by simply collapsing it to κ :

Forcing Representability 5.8 *Any set can be made representable by forcing.*

Proof: Fix any set a . First make κ indestructible. Then, collapse cardinals to κ so that $a \in H(\kappa^+)$. By the Representation Theorem, this makes a representable. \square_{Theorem}

Because cardinals are collapsed, this proof may be unsatisfying. One easy improvement is to realize that if $\alpha < \aleph_\alpha$, I can make \aleph_α representable by collapsing only α to κ . Iterating this, I can make \aleph_{\aleph_α} representable by collapsing only α . But still cardinals are collapsed. The next theorem shows how to avoid this, and add *any* set, while collapsing no cardinals above κ , to the collection of *eventually* representable sets.

Forcing Eventual Representability 5.9 *If κ is supercompact, then any set can be made eventually representable by forcing which preserves the supercompactness of κ , does not collapse cardinals above κ , and preserves all previously representable and eventually representable sets.*

Proof: First, using Exact Preservation Theorem 4.3, I may assume that the supercompactness of κ is indestructible by $<\kappa$ -directed closed forcing notions which collapse neither cardinals nor cofinalities. Since the forcing to accomplish this was a partial Laver preparation, it preserves the supercompactness of κ and admits a gap below κ . Thus, by the Enduring Representability Theorem 5.7, it preserves representability. Also, it collapses no cardinals above κ . Now I will make the set a easily definable by coding it into the continuum function above κ . The usual way of doing this, however, collapses cardinals in the case that the GCH fails, but I need not worry, since by [Sol74] the SCH holds above any supercompact cardinal, and this will be enough for my argument. By the SCH, if λ is a singular strong limit above κ it follows that $2^\lambda = \lambda^+$. I may, therefore, add subsets to λ^+ without collapsing cardinals. I may assume that a is a set of ordinals below some δ (to make decoding easier, I may even assume by further coding that a consists entirely of successor ordinals, except for its maximum element). Let $\langle \lambda_\alpha \mid \alpha < \delta \rangle$ enumerate the first δ many singular strong limits above κ . I will design a forcing notion which will ensure that $2^{\lambda_\alpha^+} = \aleph_\beta$ for β even or odd, respectively, according to whether $\alpha \in a$ or not. Let \mathbb{P} be the reverse Easton support iteration which at stage λ_α forces to add $\aleph_{\beta+1}$ many subsets to λ^+ , if β was even and I want it to be odd or vice versa. By a Δ -system argument (and this is where I use that $2^{\lambda_\alpha} = \lambda_\alpha^+$) the stage λ_α forcing $\mathbb{Q}_{\lambda_\alpha}$ is λ_α^{++} -c.c., and, since it is also $\leq \lambda_\alpha$ -closed, each stage of this iteration preserves all cardinals and cofinalities. Next, a factor argument like that in Theorem 4.3 establishes, since the SCH holds above κ , that the entire iteration preserves all cardinals and cofinalities. This forcing makes the set a concretely definable from the continuum function. Let $f(\gamma)$ be the set that is obtained by running the decoding of this information for the singular strong limits above γ . Then, if $j : V \rightarrow M$ is a θ -supercompact embedding and θ is large enough so that M has the same continuum function as V as high as any coding that I performed, then $j(f)(\kappa)$ will perform the same decoding in M as I did in V , and hence $j(f)(\kappa) = a$. Thus, f eventually represents the set a . This forcing preserves all previously (eventually) representable sets because it admits a gap below κ while preserving the supercompactness of

κ . \square_{Theorem}

Forcing Frequent Representability Theorem 5.10 *If κ is supercompact, then any class of cardinals can be made frequently representable by forcing which preserves the supercompactness of κ , does not collapse cardinals above κ , and preserves all previously representable and eventually representable sets and classes.*

Proof: Just perform the same coding as in the previous theorem. This time, however, one cannot get above all the coding. Rather, one can find arbitrarily high θ which are closure points of the coding in the sense that $A \cap \theta = A \cap H(\theta^+)$ is coded below θ . So in this case I conclude only frequent representability. \square_{Theorem}

Forcing Cardinal Representability Theorem 5.11 *Assume the SCH holds, $2^\kappa = \kappa^+$, and $\gamma^\kappa = \gamma$, where κ is supercompact. Then γ can be made representable without collapsing cardinals.*

Proof: Using Theorem 4.3, I can ensure that the supercompactness of κ is preserved by any $<\kappa$ -directed closed forcing which collapses neither cardinals nor cofinalities. And this can be done while collapsing neither cardinals nor cofinalities. Now simply force $2^\kappa = \gamma$, making γ representable by the Representation Theorem. The preparatory forcing ensures that this will preserve the supercompactness of κ , and neither cardinals nor cofinalities are collapsed. \square_{Theorem}

Corollary 5.12 *If $2^\kappa = \kappa^+$, where κ is supercompact, and $\gamma^\kappa = \gamma$, then γ can be made representable by forcing which does not collapse cardinals above κ .*

Proof: The same argument as the previous theorem. We don't need the SCH if we don't mind collapsing a few cardinals below κ . $\square_{\text{Corollary}}$

Question 5.13 *Assume the GCH. Can any set can be made representable by forcing which does not collapse cardinals?*

Perhaps this question will be answered by first making the desired set definable. The standard trick of coding the set into the continuum function—one makes the GCH hold or fail at successive cardinals in such a way so as to code the given set—does not collapse cardinals or cofinalities when the GCH holds; my proof of the Forcing Eventual Representability Theorem 5.9 shows that this can be done assuming only the SCH holds. A very interesting open question is whether there is some clever way of showing just in ZFC that any set can be made definable without collapsing cardinals or cofinalities. The difficulty, however, with this whole approach as an attack on Question 5.13 is that it is not clear that definability will give representability, even if there is some simple coding involved, since in a sense representability is local—we seem to need the information about $a \cap H(\theta^+)$ to be coded into $H(\theta^+)$, so that a θ -supercompact embedding has access to it. This is

why the Forcing Representability theorems conclude only the eventual or frequent representability of a . Fortunately for the results of this paper, this amount of representability goes a long way.

§6 Separating the Superdestructibility Hierarchy

While the levels of the supercompactness hierarchy become steadily stronger as one moves upward, this is not true of the superdestructibility hierarchy (remember that a cardinal κ is superdestructible at θ if any $<\kappa$ -closed forcing which adds a subset to θ destroys the θ -supercompactness of κ). The essential reason for this is that, with a larger θ , superdestructibility requires a stronger property to be destroyed by a larger class of forcing notions. So there is no clear implication either upwards or downwards. In fact, as I will prove in this section, I can turn superdestructibility on, then off, and then on again up through the hierarchy, in almost any conceivable pattern, making, for example, a supercompact cardinal κ superdestructible at κ^+ and κ^{+++} but not at κ^{++} or κ^{++++} , and so on. Let me begin by separating just two levels of superdestructibility.

Separation Theorem 6.1 *Suppose that $\kappa \leq \lambda < \theta$, where κ is a supercompact cardinal, λ and θ are regular, and the GCH holds. Then, while collapsing neither cardinals nor cofinalities above κ , one can make κ superdestructible at λ but not at θ , and, vice versa, superdestructible at θ but not at λ .*

Proof: I may assume, by the Forcing Cardinal Representability Theorem 5.11, that λ is represented by some function $f : \kappa \rightarrow \kappa$. Let me point out that I will not actually use much of the GCH. So far I have only used that $2^\kappa = \kappa^+$ and $\lambda^\kappa = \lambda$, when $\lambda > \kappa$, to make λ representable. And this may even destroy the GCH at κ . Later, to build the second model, I will use that $2^\lambda \leq \theta$, and that is all of the GCH that I will assume. In the two constructions below I will actually prove much more than I claimed. Suppose that ℓ is a Laver function.

The First Model. *There is a forcing extension over which the supercompactness of κ is preserved by the $<\kappa$ -directed closed posets which do not add a subset to λ , and over which the λ -supercompactness of κ is destroyed by those which do.*

Proof: Suppose that G is V -generic for the partial Laver preparation \mathbb{P} of κ in which stage γ is allowed provided that it does not add subsets to $f(\gamma)$. By the Partial Laver Preparation Theorem, κ remains supercompact in $V[G]$. The usual argument shows that the supercompactness of κ is preserved by any $<\kappa$ -directed closed forcing which does not add subsets to λ .

Let me now show that in $V[G]$, the λ -supercompactness of κ is destroyed by any $<\kappa$ -closed forcing which adds a subset to λ . That is, I will show that κ is superdestructible at λ in $V[G]$. If this is not true, then there is some $V[G]$ -generic $H \subseteq \mathbb{Q}$, where \mathbb{Q} is $<\kappa$ -closed, which adds a new set $B \subseteq \lambda$, such that κ remains λ -supercompact in $V[G][H]$. Let $j : V[G][H] \rightarrow M[j(G)][j(H)]$ be a λ -supercompact embedding. By the Enduring Representability Theorem 5.7, I know $j(f)(\kappa) = \lambda$. Since, by the closure of the embedding, $B \in M[j(G)][j(H)]$, it follows that $B \in M[j(G)]$ by the closure of the $j(\mathbb{Q})$ forcing. Also, the stage κ forcing in $j(G)$ is not allowed to add new subsets to $j(f)(\kappa) = \lambda$, and I know the stages after κ do not begin until after λ , and consequently they are $\leq \lambda$ -closed. It follows that $B \in M[G]$, and so, by the Gap Forcing Theorem 1.5, $B \in V[G]$, contradicting that B was newly added by H . $\square_{\text{First Model}}$

The Second Model. *There is a forcing extension in which κ remains supercompact and the λ -supercompactness of κ becomes fully indestructible by any $<\kappa$ -directed closed forcing, but over which any $<\kappa$ -closed forcing which adds a subset to any $\theta \geq 2^\lambda$ will destroy the θ -supercompactness of κ .*

Proof: I may assume by the High Jump Theorem 3.7 that there is a high jump function h for κ . Let \mathbb{P} be the partial Laver preparation of κ in which stage γ is allowed provided that, first, it destroys the $f(\gamma)$ -supercompactness of γ , and, second, γ is a closure point of the functions f and h . Suppose that G is V -generic for \mathbb{P} . I will show that $V[G]$ has the desired properties. By the Partial Laver Preparation Lemma 2.1, I know that κ remains supercompact in $V[G]$. And furthermore, neither cardinals nor cofinalities above κ are collapsed, because the forcing \mathbb{P} is κ -c.c.

Let me now prove that the λ -supercompactness of κ is fully indestructible in $V[G]$. Suppose that $H \subseteq \mathbb{Q}$ is $V[G]$ -generic for the $<\kappa$ -directed closed forcing $\mathbb{Q} = \dot{\mathbb{Q}}_G$, and, towards a contradiction, that κ is not λ -supercompact in $V[G][H]$. Fix $\delta \gg \lambda$ and a δ -supercompact embedding $j : V \rightarrow M$ such that $j(\ell)(\kappa) = \dot{\mathbb{Q}}$ and $\text{dom}(j(\ell)) \cap (\kappa, \delta] = \emptyset$. Notice that the stage κ forcing in $j(\mathbb{P})$ will be $\dot{\mathbb{Q}}$, provided that κ is allowed. And, since $G * H$ is M -generic for $\mathbb{P} * \dot{\mathbb{Q}}$, I know that κ will be allowed if it is not λ -supercompact in $M[G][H]$, since by representability $j(f)(\kappa) = \lambda$. But since I assumed that κ was not λ -supercompact in $V[G][H]$, it follows that κ is not λ -supercompact in $M[G][H]$, since M and V agree up to δ . Thus, the stage κ forcing in $j(\mathbb{P})$ is allowed, and so $j(\mathbb{P})$ factors as $\mathbb{P} * \dot{\mathbb{Q}} * \mathbb{P}_{\text{tail}}$, where \mathbb{P}_{tail} is $\leq \delta$ -closed, since the high jump function jumps over δ . I can therefore proceed as usual to lift the embedding to $j : V[G][H] \rightarrow M[j(G)][j(H)]$ by forcing to add G_{tail} and $j(H)$. Again, using $j \restriction \lambda$ as a seed, I generate a normal fine measure on

$P_\kappa\lambda$ which can not have been added by the tail forcing, and consequently must be in $V[G][H]$, contradicting our assumption that κ was not λ -supercompact there.

Finally, let me prove that any $<\kappa$ -closed forcing which adds a subset to $\theta \geq 2^\lambda = 2^{\lambda^{<\kappa}}$ will destroy the θ -supercompactness of κ . Suppose towards a contradiction that κ remains θ -supercompact in $V[G][H]$, where $H \subseteq \mathbb{Q}$ is $V[G]$ -generic and adds a subset $B \subseteq \theta$. Then there is a θ -supercompact embedding $j : V[G][H] \rightarrow M[j(G)][j(H)]$. By the Enduring Representability Theorem 5.7, I know $j(f)(\kappa) = \lambda$. If κ is allowed, then κ is not λ -supercompact in $M[G][g]$, where g is the stage κ generic of $j(G)$. But the additional forcing to $M[j(G)][j(H)]$ is $\leq\theta$ -closed, since $j(h)(\kappa) > \theta$, and so κ is not λ -supercompact in $M[j(G)][j(H)]$. But κ is λ -supercompact in $V[G][H]$, so by coding a measure on $P_\kappa\lambda$ with a subset of θ and using the closure of j , we see that κ is λ -supercompact in $M[j(G)][j(H)]$, a contradiction. Thus, κ must not be allowed. In this case, I know that $B \in M[G]$ by the closure of the tail forcing, and so, by the Gap Forcing Theorem 1.5, $B \in V[G]$, contradicting our assumption that B was newly added by H . So in any case I get a contradiction. $\square_{\text{Second Model}}$

This completes the proof of the theorem. \square_{Theorem}

Let me prove next a great generalization of the previous theorem.

Superdestruction Separation Theorem 6.2 *Suppose that A is a representable class of cardinals each with cofinality at least κ , and that the GCH holds. Then there is a forcing extension $V[G]$, preserving all cardinals, cofinalities, and the GCH, in which:*

1. *If $\lambda \notin A$, then κ is superdestructible at λ ; any $<\kappa$ -closed forcing which adds a subset to λ destroys the λ -supercompactness of κ .*
2. *If $\lambda \in A$, then κ is not superdestructible at λ . Indeed, any $<\kappa$ -directed closed forcing which adds no bounded sets to λ and which preserves all cardinals, cofinalities and the GCH will preserve the λ -supercompactness of κ .*

Proof: Suppose that A is represented by the function f . By the High Jump Theorem 3.7, I may assume that there is a high jump function h for κ . Let \mathbb{P} be the partial Laver preparation of κ in which stage γ is allowed provided first, that \mathbb{Q}_γ adds no bounded subsets to some $\lambda \in f(\gamma)$; second, that γ is not λ -supercompact, for this same λ , in $V^{\mathbb{P}_\gamma * \mathbb{Q}_\gamma}$; third, that \mathbb{Q}_γ preserves all cardinals, cofinalities, and the GCH; and finally, fourth, that γ is a closure point both of f and the high jump function h , in the sense that $f \restriction \gamma \subseteq V_\gamma$ and $h \restriction \gamma \subseteq \gamma$. This defines the forcing \mathbb{P} . Now suppose that $G \subseteq \mathbb{P}$ is V -generic. Let me prove that $V[G]$ has the desired

properties. I know by the Partial Laver Preparation Lemma 2.1 that κ remains supercompact in $V[G]$. Furthermore, by the argument of Theorem 4.3, \mathbb{P} preserves all cardinals, cofinalities, and the GCH.

Let me prove that 1 holds. Suppose λ is not in A , but that κ remains λ -supercompact in $V[G][H]$ where $H \subseteq \mathbb{Q}$ is $V[G]$ -generic for the $<\kappa$ -closed forcing \mathbb{Q} which adds a subset $B \subseteq \lambda$. Then there must be a λ -supercompact embedding $j : V[G][H] \rightarrow M[j(G)][j(H)]$ witnessing this. Necessarily, $B \in M[j(G)][j(H)]$. Because I only forced at stages which were closure points of the high jump function, it follows that there is no forcing in $j(\mathbb{P})$ in the interval $(\kappa, \lambda]$. Suppose, momentarily, that there is no forcing at stage κ in $j(\mathbb{P})$. In this case, $j(G) = G * G_{\text{tail}}$ and by the previous remarks G_{tail} is $\leq \lambda$ -closed. Consequently, by closure considerations, $B \in M[G]$, and since $M \subseteq V$ by the Gap Forcing Theorem 1.5 it follows that $B \in V[G]$, a contradiction. Thus there must be forcing at stage κ . In this case $j(G) = G * g * G_{\text{tail}}$, where $g \subseteq \mathbb{Q}_\kappa$ is the stage κ forcing, with the corresponding $\lambda' \in j(f)(\kappa)$ to which g adds no bounded sets, and such that κ is not λ' -supercompact in $M[G][g]$. Since every element of A has cofinality at least κ , I may assume $\text{cof}(\lambda') \geq \kappa$. Again I know that G_{tail} is $\leq \lambda$ -closed. Therefore $B \in M[G][g]$. Also, since g was allowed I know that it preserved all cardinals, cofinalities, and the GCH. By the closure of the forcing and of j I know that $P(\lambda)^{M[G][g]} = P(\lambda)^{M[j(G)][j(H)]} = P(\lambda)^{V[G][H]}$. Thus, by coding measures from $V[G][H]$ on $P_\kappa \beta$ into subsets of λ , I conclude that κ is β -supercompact in $M[G][g]$ for every $\beta < \lambda$ with $\text{cof}(\beta) \geq \kappa$. It follows that $\lambda \leq \lambda'$. Since $\lambda \notin A$ and $\lambda' \in j(f)(\kappa)$ it follows that $\lambda \neq \lambda'$, since $j(f)(\kappa)$ and A agree up to λ^+ . Consequently $\lambda < \lambda'$. Thus, since g adds no bounded sets to λ' , I conclude that $B \in M[G]$, and, as in the first case, that $B \in V[G]$, a contradiction. So 1 is proved.

To prove 2, suppose that $\lambda \in A$ and that $H \subseteq \mathbb{Q}$ is $V[G]$ -generic, where \mathbb{Q} is $<\kappa$ -closed, adds no bounded sets to λ , and preserves all cardinals, cofinalities, and the GCH. Towards a contradiction, suppose that κ is not λ -supercompact in $V[G][H]$. Fix $\theta \gg \lambda$ and, in V , a θ -supercompact embedding $j : V \rightarrow M$, such that $j(\ell)(\kappa) = \dot{\mathbb{Q}}$ and $\text{dom}(j(\ell)) \cap (\kappa, \theta] = \emptyset$. Since κ is not λ -supercompact in $V[G][H]$, it is also not supercompact in $M[G][H]$, and so κ is an allowed stage of $j(\mathbb{P})$. I may therefore employ the usual lifting argument to lift to $j : V[G][H] \rightarrow M[j(G)][j(H)]$ in a forcing extension $V[G][H][G_{\text{tail}}][j(H)]$. As usual, using $j \restriction \lambda$ as a seed, I obtain a measure μ on $P_\kappa \lambda$ which must be in $V[G][H]$, contrary to our assumption. \square_{Theorem}

§7 Exact Preservation As You Like It

In the previous sections we discovered a few major landmarks in the unexplored region between indestructibility and superdestructibility. In this section I will point out a great mountain range, spanning the continent. Specifically, after proving the Exact Preservation Theorems, more powerful than the ones in §3, my arguments will culminate in the ‘As You Like It’ Theorem, the title theorem of this paper. Recall that a set of ordinals is *fresh* over V when every proper initial segment of it is in V , but the set itself is not in V . Let us begin.

Exact Preservation 7.1 *Suppose that κ is supercompact in V and that A is a class of cardinals. Then there is a forcing extension $V[G]$, obtained without collapsing cardinals above κ , in which κ remains supercompact and over which the supercompactness of κ is preserved by exactly those $<\kappa$ -directed closed posets which collapse no cardinal in A .*

Proof: Again by forcing if necessary I may assume that A is frequently represented by some function f , and that there is a high jump function h . Now suppose G is V -generic for the partial Laver preparation of κ in which γ is allowed if \mathbb{Q}_γ collapses no cardinal in $f(\gamma)$ and γ is closed under the high jump function. The usual arguments establish that the supercompactness of κ in $V[G]$ is preserved by any forcing which collapses no cardinals in A . Let me now prove it is destroyed by those which do. Suppose $H \subseteq \mathbb{Q}$ is $V[G]$ -generic for $<\kappa$ -closed forcing \mathbb{Q} which collapses a cardinal $\lambda \in A$, but that κ remains θ -supercompact, for some large $\theta \geq \lambda$ such that f represents A with respect to the witness embedding $j : V[G][H] \rightarrow M[j(G)][j(H)]$. (By the Going-Down Lemma 5.6 I may in fact take $\theta = \lambda$ here.) The former cardinal λ must be collapsed in $M[j(G)][j(H)]$, and since the high jump function jumps over θ , it must be that λ is collapsed at stage κ in $j(G)$. But $j(f)(\kappa)$ and A agree on λ , and so the stage κ forcing would not be allowed if it collapsed λ , a contradiction. \square_{Theorem}

Exact Preservation 7.2 *Suppose that κ is supercompact in V and A is a class of cardinals. Then there is a forcing extension $V[G]$, obtained without collapsing cardinals above κ , in which κ remains supercompact and over which the supercompactness of κ is preserved by exactly those $<\kappa$ -directed closed posets which collapse a cardinal of A .*

Proof: Again, by forcing if necessary, I may assume that A is frequently represented by f , and that there is a high jump function h . For this argument, let stage γ be allowed when \mathbb{Q}_γ collapses a cardinal in $f(\gamma)$ and γ is closed under h . The

preservation arguments go through as usual. Suppose, conversely, that $H \subseteq \mathbb{Q}$ is $V[G]$ -generic for $<\kappa$ -closed forcing which collapses no cardinal of A , and that κ remains λ -supercompact for some large $\lambda \geq |\mathbb{Q}|$ such that f represents A with respect to a λ -supercompactness embedding $j : V[G][H] \rightarrow M[j(G)][j(H)]$. Since the high jump function jumps over λ , it follows that $H \in M[G][g]$, where g is the stage κ forcing (the forcing g is nontrivial since $H \notin M[G]$). Observe that $V[G][g] = V[G][H]$, and so the g forcing is equivalent over $V[G]$ to a forcing of size $\leq \lambda$. This isomorphism must lie in $M[G]$, and so g is unable to collapse cardinals above λ in $M[G]$. But g must collapse a cardinal in $j(f)(\kappa)$, since it was allowed. Thus, over $M[G]$, and hence also over $V[G]$, the generic g collapses a cardinal in A below λ . This contradicts that H collapsed no such cardinals over $V[G]$. \square_{Theorem}

Exact Preservation 7.3 *Suppose that κ is supercompact and A is a class of regular cardinals. Then there is a forcing extension $V[G]$, obtained without collapsing cardinals above κ , over which the supercompactness of κ is preserved by exactly those $<\kappa$ -directed closed posets which add a fresh subset, over $V[G]$, to a cardinal in A .*

Proof: First, by the High Jump Theorem 3.7, I may assume that there is a high jump function h for κ . Also, by forcing if necessary, I may assume that A is frequently representable, by some function f . Suppose now that G is generic for the partial Laver preparation \mathbb{P} of κ in which stage γ forcing is allowed if first, it adds a fresh subset over $V^{\mathbb{P}_\gamma}$ to some $\lambda \in f(\gamma)$, and second, γ is closed under h . I know by the Partial Laver Preparation Lemma 2.1 that κ remains supercompact in $V[G]$ and the usual preservation argument shows that the supercompactness of κ is preserved by any $<\kappa$ -directed closed poset which adds a fresh subset to an element of A .

Now suppose $H \subseteq \mathbb{Q}$ is $V[G]$ -generic for $<\kappa$ -closed \mathbb{Q} and H does not add a fresh set, over $V[G]$, to any element of A . I would like to show κ is no longer supercompact. Suppose, towards a contradiction, that $j : V[G][H] \rightarrow M[j(G)][j(H)]$ is a λ -supercompact embedding, where $|\mathbb{Q}| \leq \lambda$, and λ is such that f represents A with respect to j . Thus, $H \in M[j(G)][j(H)]$, and since the high jump function h jumps over λ , I know therefore that $H \in M[G][g]$, where g is the (perhaps trivial) stage κ forcing. Since $H \notin V[G]$, it must be also that $H \notin M[G]$ and so g is actually nontrivial. Notice that $V[G][H] = V[G][g]$, and consequently, below a condition, the g forcing is isomorphic, in $V[G]$, to the H forcing below a condition. Since this isomorphism has size less than or equal to λ , it must lie in $M[G]$, and consequently,

by the chain condition, g cannot add a fresh subset to any regular cardinal in $M[G]$ above λ . But since g was allowed, it must have added, over $M[G]$, a fresh subset $B \subseteq \zeta$ for some $\zeta \in j(f)(\kappa)$. By the previous observation it follows that $\zeta \leq \lambda$, and consequently $\zeta \in A$. Since $B \notin M[G]$ and $\zeta \leq \lambda$ it follows that $B \notin V[G]$, and so H has added, over $V[G]$, a fresh subset to an element of A , contradicting my assumption. \square_{Theorem}

Exact Preservation 7.4 *Suppose that κ is supercompact in V and that A is a frequently representable class of cardinals at which the GCH holds. Then there is a forcing extension $V[G]$, obtained without collapsing cardinals or disturbing the GCH above κ , in which κ remains supercompact, and over which the supercompactness of κ is preserved by exactly those $<\kappa$ -directed closed posets which preserve the GCH at the cardinals of A .*

Proof: Again assume A is frequently represented by f , and that there is a high jump function h . This time let the stage γ forcing be allowed when it preserves the GCH at every cardinal in $f(\gamma)$ and γ is closed under the high jump function. The usual arguments establish that the supercompactness of κ is preserved over $V[G]$ by $<\kappa$ -directed closed posets which preserve the GCH at the cardinals of A . Suppose, conversely, that $H \subseteq \mathbb{Q}$ violates the GCH at $\lambda \in A$, but that κ remains θ -supercompact with embedding $j : V[G][H] \rightarrow M[j(G)][j(H)]$, where $\theta \geq \lambda$ is such that f represents A with respect to the θ -supercompact embeddings. Because the high jump function jumps over λ , only the stage κ forcing could ruin the GCH at λ in $M[j(G)][j(H)]$, but it is in precisely this case that it is not allowed, a contradiction. \square_{Theorem}

Exact Preservation 7.5 *Suppose that κ is supercompact in V and that A is a frequently representable class of cardinals at which the GCH holds. Then there is a forcing extension $V[G]$, obtained without collapsing cardinals or disturbing the GCH above κ , in which κ remains supercompact and over which the supercompactness of κ is preserved by exactly those $<\kappa$ -directed closed posets which destroy the GCH at a cardinal of A .*

Proof: Assume A is frequently represented by f , and that h is a high jump function. Let the stage γ forcing be allowed when it destroys the GCH at some element of $f(\gamma)$ and γ is closed under the high jump function. Again the usual lifting arguments show that the supercompactness of κ in $V[G]$ is preserved by any $<\kappa$ -directed closed forcing which destroys the GCH at some element of A . Conversely, suppose $H \subseteq \mathbb{Q}$ is $V[G]$ -generic for $<\kappa$ -closed forcing H which does not destroy the GCH at any

cardinal in A , but that κ remains λ -supercompact with embedding $j : V[G][H] \rightarrow M[j(G)][j(H)]$ for some large $\lambda \geq |\mathbb{Q}|$ which works with f . Again the high jump function jumps over λ , so $H \in M[G][g]$ where g is the stage κ forcing in $j(\mathbb{P})$; since $H \notin M[G]$ this forcing is nontrivial. Observe that $V[G][g] = V[G][H]$, and so the g forcing is equivalent in $V[G]$ to forcing of size at most λ , and the isomorphism must be in $M[G]$. Thus, over $M[G]$, the generic g does not affect the GCH above λ . Since it was allowed, it must have destroyed the GCH at some element of $j(f)(\kappa)$ below λ , and since f represents A , this element must be in A . Thus, over $V[G]$, the generic g , and hence also H , destroyed the GCH at an element of A , contrary to my assumption on H . \square_{Theorem}

The previous theorems display the power of the Gap Forcing Theorem 1.5 to severely limit the sort of supercompactness embeddings which can exist in a gap forcing extension. All the Exact Preservation Theorems, however, are special cases of, and follow as immediate corollaries to, my next theorem, the ‘As You Like It’ Theorem, which asserts that one can tailor the universe, by forcing, so that nearly any desired class of posets will preserve the supercompactness of κ , and the others destroy it. It therefore encompasses all of the particular properties in the Exact Preservation Theorems, and unifies their proofs.

I will now make two key definitions. Suppose $\{\mathbb{Q} \mid \phi(\mathbb{Q}, \gamma, A, G)\}$ is a class of $<\gamma$ -directed closed posets defined using the formula ϕ and, as parameters, a cardinal γ , a class A , and a set G such that $|G| \leq \gamma$. I will say that this class, or the formula ϕ , is *local* provided that, in any model of set theory, it can be decided whether a given poset $\mathbb{Q} \in H(\lambda^+)$ is a member of the class by consulting only $H(\lambda^+)$: that is, first, the truth of $\phi(\mathbb{Q}, \gamma, A, G)$ does not depend fully on A but rather only on $A \cap H(\lambda^+)$, and furthermore, second, that the truth of $\phi(\mathbb{Q}, \gamma, A, G)$ is absolute to any other model with the same $H(\lambda^+)$. The formula ϕ , with parameters, *respects the equivalence of forcing* iff in any model of set theory, whenever $\phi(\mathbb{Q})$ holds, and \mathbb{Q} and \mathbb{Q}' have isomorphic complete boolean algebras, then $\phi(\mathbb{Q}')$ also holds; also, $\phi(\mathbb{Q})$ holds just in case for densely many $b \in \mathbb{Q}$, $\phi(\mathbb{Q}_b)$ holds, where \mathbb{Q}_b denotes the part of the poset \mathbb{Q} below the condition b . It follows that $\phi(\mathbb{Q})$ holds just in case $\phi(\mathbb{Q}_b)$ holds for every $b \in \mathbb{Q}$. For example, the formulas “ \mathbb{Q} preserves every cardinal in A ,” “ \mathbb{Q} preserves the GCH at the cardinals of A ,” and “ \mathbb{Q} adds a fresh subset to an element of A ” are all local definitions which respect the equivalence of forcing: if $\mathbb{Q} \in H(\lambda^+)$, then, since all the relevant names in question, for the collapsing functions or the fresh sets, are also in $H(\lambda^+)$, it follows that any other

model with the same $H(\lambda^+)$ will agree on $\phi(\mathbb{Q})$.

The ‘As You Like It’ Theorem 7.6 *The class of $<\kappa$ -directed closed posets which preserve the supercompactness of κ can be made by forcing to be defined by any pre-selected local formula which respects the equivalence of forcing.*

More precisely: suppose that κ is supercompact in V and that ϕ is any local formula you like, with class parameter A , which respects the equivalence of forcing. Then one can force to a model $V[G]$ where κ remains supercompact, and where, for any $<\kappa$ -directed closed poset \mathbb{Q} in $V[G]$:

1. *If $\phi(\mathbb{Q}, \kappa, A, G)$ holds, then \mathbb{Q} preserves the supercompactness of κ .*
2. *If $\phi(\mathbb{Q}, \kappa, A, G)$ fails, then, below a condition, \mathbb{Q} destroys the supercompactness of κ .*

Proof: Assume that κ is supercompact. I may, by the High Jump Theorem 3.7, suppose also that there is a high jump function h for κ . Furthermore, by forcing if necessary, I may assume that A is frequently represented by some function f . Let ℓ be a Laver function. I may assume that every point in $\text{dom}(\ell)$ is a closure point of the high jump function h , and also of f . The forcing \mathbb{P} will be the partial Laver preparation of κ in which, at the very first stage, to avoid triviality, I add a Cohen real, and then, at subsequent stages γ , the forcing $\mathbb{Q}_\gamma = \ell(\gamma)_{G_\gamma}$ is allowed provided that $V[G_\gamma] \models \phi(\mathbb{Q}_\gamma, \gamma, f(\gamma), G_\gamma)$. Thus, after the first stage, I perform the Laver preparation exactly when the Laver function ℓ hands me a poset which satisfies the formula ϕ in the appropriate model.

Suppose now that $G \subseteq \mathbb{P}$ is V -generic, and that \mathbb{Q} is a $<\kappa$ -directed closed poset in $V[G]$. I know that κ is supercompact in $V[G]$ by the Partial Laver Preparation Lemma 2.1. It remains to prove the other two properties.

Lemma 7.6.A *If $\phi(\mathbb{Q}, \kappa, A, G)$ holds in $V[G]$, then \mathbb{Q} preserves the supercompactness of κ .*

Proof: As in the previous theorems, the usual Laver argument adapts to this circumstance. Suppose $H \subseteq \mathbb{Q}$ is $V[G]$ -generic. Fix $\lambda \geq |\mathbb{Q}|$ and $\theta \gg \lambda$, and let $j : V \rightarrow M$ be a θ -supercompact embedding such that $j(\ell)(\kappa) = \dot{\mathbb{Q}}$ and $\text{dom}(j(\ell)) \cap (\kappa, \theta] = \emptyset$. I must argue that the stage κ forcing is allowed. Since $\phi(\mathbb{Q}, \kappa, A, G)$ holds in $V[G]$, it also holds in $M[G]$, since ϕ is local and $H(\lambda^+)^{V[G]} = H(\lambda^+)^{M[G]}$. Also, by representability, $A \cap H(\theta^+) = j(f)(\kappa) \cap H(\theta^+)$, and, since ϕ depends not fully on A but only on $A \cap H(\theta^+)$, it follows that $\phi(\mathbb{Q}, \kappa, j(f)(\kappa), G)$ holds in $M[G]$. Thus, the stage κ forcing is allowed. The forcing $j(\mathbb{P})$, therefore, factors as $\mathbb{P} * \mathbb{Q} * \mathbb{P}_{\text{tail}}$, where \mathbb{P}_{tail} is $\leq \theta$ -closed in $M[G][H]$, and hence also in $V[G][H]$. Thus,

as usual, I can force to add $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ generic over $V[G][H]$, and lift the embedding to $j : V[G] \rightarrow M[j(G)]$ where $j(G) = G * H * G_{\text{tail}}$. After this, I can also lift the embedding through the H -forcing, using the master condition $j \restriction H$, and the fact that $j(\mathbb{Q})$ is $< j(\kappa)$ -directed closed. Adding a further generic $j(H) \subseteq j(\mathbb{Q})$, I lift to $j : V[G][H] \rightarrow M[j(G)][j(H)]$. Again, using $j \restriction \lambda$ as a seed, I generate a normal fine measure μ on $P_\kappa \lambda$. The measure μ cannot have been added by G_{tail} or by $j(H)$, and so μ is in $V[G][H]$, witnessing that κ is λ -supercompact there. \square_{Lemma}

Lemma 7.6.B *If $\phi(\mathbb{Q}, \kappa, A, G)$ fails in $V[G]$, then, below a condition, \mathbb{Q} destroys the supercompactness of κ .*

Proof: Let me prove the contrapositive. Suppose that every generic extension by \mathbb{Q} preserves the supercompactness of κ . Since ϕ respects the equivalence of forcing, I have merely to show that for densely many $b \in \mathbb{Q}$ the relation $\phi(\mathbb{Q}_b, \kappa, A, G)$ holds in $V[G]$. Fix any $b' \in \mathbb{Q}$; I intend to find a $b \leq b'$ such that $\phi(\mathbb{Q}_b, \kappa, A, G)$ holds in $V[G]$. Suppose that $H \subseteq \mathbb{Q}$ is $V[G]$ -generic below b' , where \mathbb{Q} is coded by some subset of λ . Suppose that κ is still θ -supercompact in $V[G][H]$, where $\theta \geq \lambda$ is such that f represents A with respect to a θ -supercompact embedding $j : V[G][H] \rightarrow M[j(G)][j(H)]$. (By the Going-Down Lemma 5.6, I may in fact take $\theta = \lambda$ here.) It follows that $H \in M[j(G)][j(H)]$. Factor $j(\mathbb{P})$ as $\mathbb{P} * \tilde{\mathbb{Q}} * \mathbb{P}_{\text{tail}}$ and $j(G)$ as $G * g * G_{\text{tail}}$, where $g \subseteq \tilde{\mathbb{Q}}$ is the (possibly trivial) stage κ forcing in $j(\mathbb{P})$. Since h jumps over θ , I know that the next forcing cannot occur until past θ , so \mathbb{P}_{tail} is $\leq \theta$ -closed. Thus, $H \in M[G][g]$, and so, by the Gap Forcing Theorem 1.5, since $M \subseteq V$ it follows also that $H \in V[G][g]$. But $g \in V[G][H]$, and so $V[G][g] = V[G][H]$. Thus, the forcing $\tilde{\mathbb{Q}}$ and the forcing \mathbb{Q} produce the same generic extension over $V[G]$. It follows that $\text{ro}(\mathbb{Q}_b) \cong \text{ro}(\tilde{\mathbb{Q}}_c)$ for some conditions $b \in \mathbb{Q}$ and $c \in \tilde{\mathbb{Q}}$. I may assume $b \leq b'$. Observe that $\phi(\tilde{\mathbb{Q}}, \kappa, A, G)$ holds in $M[G]$. By the Gap Forcing Theorem, $M[G]$ and $V[G]$ have the same $H(\theta^+)$. Therefore, since ϕ is local, $\phi(\tilde{\mathbb{Q}}, \kappa, A, G)$ also holds in $V[G]$. It follows, since ϕ respects the equivalence of forcing, that $\phi(\tilde{\mathbb{Q}}_c, \kappa, A, G)$ holds in $V[G]$, and hence also that $\phi(\mathbb{Q}_b, \kappa, A, G)$ holds there, as desired. \square_{Lemma}

Thus, $V[G]$ is as required. \square_{Theorem}

Remark on Closure 7.7 Again let me point out that directed closure is only needed on the preservation side, to find a master condition. On the destruction side, it is enough to assume that \mathbb{Q} is $< \kappa$ -strategically closed.

§8 Epilogue: Fragility \perp Superdestructibility

At first glance, fragility and superdestructibility seem to be made of the same delicate material. But this is not so. In this epilogue, I will show that neither property implies the other, and I will construct models which exhibit each of the four possibilities.

The notion of fragility first appeared in my first paper [Ham94a], and subsequently in my dissertation [Ham94b], where I defined that a large cardinal κ is *fragile* when any forcing which preserves κ^+ and $2^{<\kappa}$ and adds a subset to κ destroys the measurability of κ . The notion of superdestructibility appeared first in [Ham97b], where I defined that a large cardinal κ is *superdestructible* when any $<\kappa$ -closed forcing which adds a subset of κ destroys the measurability of κ . I will now prove that these notions, though similar, are actually independent.

Fragility \perp Superdestructibility Theorem 8.1 *Suppose κ is a supercompact cardinal in V . Then in various forcing extensions where κ remains supercompact,*

1. *κ is both fragile and superdestructible.*
2. *κ is fragile, but not superdestructible.*
3. *κ is superdestructible, but not fragile.*
4. *κ is neither fragile nor superdestructible.*

I will prove each of the four possibilities separately. So, for the remainder of this paper, assume that κ is a supercompact cardinal in V . For the first possibility, I will also show the surprising fact that one can obtain a model in which κ is fragile, superdestructible, and, simultaneously, indestructible above κ —the supercompactness of κ is preserved by any $<\kappa$ -directed closed poset which adds no subsets to κ .

Possibility One. *There is a forcing extension in which κ is fragile, superdestructible, and, simultaneously, indestructible above κ .*

Proof: In fact, the fragile measurability models of [Ham94a] also have superdestructibility. In order to get indestructibility above κ , I will introduce here a wrinkle to the construction in the Fragile Measurability Theorem 3.12 of [Ham94a]. While familiarity with that argument will ease comprehension of this one, I aim to give here a complete, if terse, presentation.

So, suppose that κ is supercompact in V . I may assume, by forcing if necessary, that $V \models \text{GCH}$. Let ℓ be a Laver function for κ . Our forcing $\mathbb{P}_{\kappa+1} = \mathbb{P}_\kappa * \mathbb{Q}_\kappa$ will be a reverse Easton $(\kappa + 1)$ -iteration with nontrivial forcing only at inaccessible

stages $\gamma \in \text{dom}(\ell)$. The forcing \mathbb{Q}_γ at stage γ will be one of three types. First, the Laver function ℓ might instruct us to perform what I will call the fragility forcing at stage γ . In this case, $\ell(\gamma)$ will hand us a pair $\langle a_\gamma, d_\gamma \rangle$, called γ -data packets in [Ham94a], such that $a_\gamma = \langle \dot{a}_\gamma^\alpha \mid \alpha < \gamma^+ \rangle$ enumerates some relations on γ such that $\text{ot}(\gamma, \dot{a}_\gamma^\alpha) = \alpha$ for every $\alpha < \gamma^+$, and $d_\gamma = \langle D_\gamma^\alpha \mid \alpha < \gamma^+ \rangle$ enumerates $P(\gamma)^V$. If there is fragility forcing at some stage $\delta < \gamma$, then it will have added a sequence $\langle C_\delta^\alpha \mid \alpha < \delta^+ \rangle$ of club subsets of δ , so I may refer to these clubs when defining \mathbb{Q}_γ . Let \mathbb{Q}_γ^α be the poset which adds, with conditions which are initial segments, a club set $C_\gamma^\alpha \subseteq \gamma$ with the property that if δ is an inaccessible cluster point of C_γ^α , then $\langle \gamma, \alpha \rangle$ reflects to $\langle \delta, \alpha' \rangle$ for some α' in the sense that first of all there was fragility forcing at stage δ , but secondly the data packets and clubs agree:

$$\dot{a}_\gamma^\alpha \restriction \delta = \dot{a}_\delta^{\alpha'} \quad D_\gamma^\alpha \cap \delta = D_\delta^{\alpha'} \quad C_\gamma^\alpha \cap \delta = C_\delta^{\alpha'}.$$

Let \mathbb{Q}_γ be the $<\gamma$ -support product $\prod \mathbb{Q}_\gamma^\alpha$. By a Δ -system argument, this is γ^+ -c.c. This defines the fragility type forcing at stage γ .

The Laver function ℓ , secondly, may instruct us to perform Laver preparation forcing at stage γ . In this case, $\ell(\gamma)$ will hand us a $<\gamma$ -directed closed poset \mathbb{Q}_γ , which will be our stage γ forcing provided that, additionally, it adds no new subsets to γ .

Finally, third, the Laver function may instruct us to perform both kinds of forcing. In this case, $\ell(\gamma)$ will hand us both the γ -data packet, and also the (name of) a $<\gamma$ -directed closed poset which adds no subsets to γ . The stage γ forcing will consist of first performing the fragility forcing and then the indestructibility forcing.

This completely describes the iteration \mathbb{P}_κ . The forcing \mathbb{Q}_κ at stage κ will be of the fragility forcing type, with some specific κ -data packet $\langle a_\kappa, d_\kappa \rangle$. Let $G * g \subseteq \mathbb{P}_\kappa * \mathbb{Q}_\kappa$ be V -generic for this forcing. Thus $g = \langle C_\kappa^\alpha \mid \alpha < \kappa^+ \rangle$ is a sequence of club subsets of κ with the reflection property. Let me now prove that $V[G][g]$ has the desired properties. First observe that if there is forcing at stage γ , then γ^+ is not collapsed.

Let me now prove that κ is fragile in $V[G][g]$. Suppose towards a contradiction that $H \subseteq \mathbb{Q}$ is generic, adds a set $A \subseteq \kappa$, preserves $\kappa^{<\kappa}$ and κ^+ , but that κ remains measurable in $V[G][g][H]$. Thus, there is an embedding $j : V[G][g][H] \rightarrow M[j(G)][j(g)][j(H)]$. Since \mathbb{Q} is not necessarily closed, the Gap Forcing Theorem 1.5 does not apply, but the set A must be in $M[j(G)]$ since it can be added by neither $j(g)$ nor $j(H)$. Also, since κ is necessarily an inaccessible closure point of the clubs in $j(g)$, it follows that $j(G)$ must have performed fragility forcing at stage

κ . Moreover, by the reflection property of the clubs in $j(g)$, since κ is an inaccessible cluster point of $j(C_\kappa^\alpha)$, it follows that the clubs added by the forcing at stage κ in $j(G)$ are $j(C_\kappa^\alpha) \cap \kappa = C_\kappa^\alpha$. What is more, since $j(\triangleleft_\kappa^\alpha) \restriction \kappa = \triangleleft_\kappa^\alpha$, it follows that C_κ^α is the generic used in the α^{th} coordinate in the fragility forcing at stage κ in $j(\mathbb{P})$. Since κ^+ is preserved, this means that the stage κ fragility forcing in $j(G)$ is actually g . Thus, it follows that $j(G) = G * g * h * g_{\text{tail}}$, where h is the stage κ indestructibility forcing, if it exists, or $j(G) = G * g * g_{\text{tail}}$, if there is no stage κ indestructibility forcing. In either case, since the indestructibility forcing h is not allowed to add subsets to κ , the set A must be in $M[G][g]$. Since A is a subset of κ , I know moreover that $A \in M[G][g \restriction \alpha]$ for some $\alpha < \kappa^+$, and so $A = \dot{A}_{G * g \restriction \alpha}$, for some name $\dot{A} \in M$. Because of the reflection $j(D_\kappa^\alpha) \cap \kappa = (D_\kappa^\alpha)^M$ I know that $P(\kappa)^M = P(\kappa)^V$, since these are both enumerated by d_κ , and thus $\dot{A} \in V$. Hence, $A \in V[G][g]$ contrary to our assumption that A was new.

Next, I will prove that κ is superdestructible in $V[G][g]$. Suppose that $H \subseteq \mathbb{Q}$ is generic, where \mathbb{Q} is $<\kappa$ -closed and H adds a new set $A \subseteq \kappa$, but that κ is still measurable in $V[G][g][H]$. Thus, there is an embedding $j : V[G][g][H] \rightarrow M[j(G)][j(g)][j(H)]$. As above, I know that $A \in M[j(G)]$ by closure considerations. I do not, however, know so easily that the stage κ forcing of $j(G)$ is g , since it may be that \mathbb{Q} collapsed κ^+ . Nevertheless, since each \mathbb{Q}_γ has dense sets as closed as you like up to γ , it follows that $\mathbb{P} * \mathbb{Q}$ admits a gap below κ , and thus, by the Gap Forcing Theorem 1.5, $P(\kappa)^M = P(\kappa)^V$, and consequently $\kappa^{+M} = \kappa^{+V}$. In fact,

$$\kappa^{+V} = \kappa^{+M} = \kappa^{+M[j(G)]} = \kappa^{+M[j(G)][j(g)][j(H)]} = \kappa^{+V[G][g][H]}.$$

The first equality holds as I explained just now. The second holds by my remark that the successor cardinals of nontrivial forcing stages are not collapsed. The third holds by the closure of the forcing. And the fourth holds by the closure of the embedding. Thus, in fact, \mathbb{Q} did not collapse κ^+ . Since \mathbb{Q} also preserves $2^{<\kappa}$ and adds a subset to κ , it follows from the already established fact that κ is fragile that \mathbb{Q} destroys the measurability of κ .

Finally, I will argue that κ is indestructible above κ in $V[G][g]$. This argument will also establish that κ is supercompact in $V[G][g]$. Suppose $H \subseteq \mathbb{Q}$ is generic, where \mathbb{Q} is $<\kappa$ -directed closed and does not add new subsets to κ . Fix any λ and let $\theta \gg \lambda$. Fix $j : V \rightarrow M$ a θ -supercompact embedding in V such that $j(\ell)(\kappa)$ instructs us to first perform the composite forcing at stage κ —first the fragility forcing and then \mathbb{Q} —and such that $\text{dom}(j(\ell)) \cap (\kappa, \theta] = \emptyset$. I will lift j to $V[G][g][H]$. First, I can lift to $j : V[G] \rightarrow M[j(G)]$ by using $g * H$ as the stage κ generic, and then forcing to

add a tail G_{tail} . Now I have to construct a master condition below $j \restriction \kappa^+$ as in [Ham94a], and force below it to lift to $j : V[G][g] \rightarrow M[j(G)][j(g)]$. The master condition is simply the condition $p \in j(\mathbb{Q}_\kappa)$, with support $j \restriction \kappa^+$, such that $p(j(\alpha)) = \bar{C}_\kappa^\alpha$, where $\bar{C}_\kappa^\alpha = C_\kappa^\alpha \cup \{\kappa\}$. Below this condition there is in $j(\mathbb{Q}_\kappa)$ a dense set which is $\leq \theta$ -closed, namely, the set of conditions q which mention a point above θ on every coordinate in their support. So I can, by forcing over $j(\mathbb{Q}_\kappa)$ below the master condition, lift j through the \mathbb{Q}_κ forcing. Next, use the directed closure of $j(\mathbb{Q})$ to find a master condition below $j \restriction H$, and lift fully to $j : V[G][g][H] \rightarrow M[j(G)][j(g)][j(H)]$. This embedding lives in $V[G][g][H][G_{\text{tail}}][j(g)][j(H)]$. But using $j \restriction \lambda$ as a seed, I conclude that there is a measure witnessing λ -supercompactness which could not have been added by the tail forcing $G_{\text{tail}} * j(g) * j(H)$. So the measure lives in $V[G][g][H]$, and so κ is λ -supercompact there, as desired. In the case that \mathbb{Q} is trivial, we conclude also that κ is supercompact in $V[G][g]$. This completes the proof. $\square_{\text{Possibility One}}$

Possibility Two. *There is a forcing extension in which κ is fragile, but not superdestructible. In fact, κ can be made simultaneously fragile and indestructible by any $< \kappa$ -directed closed forcing which collapses κ^+ .*

Proof: Here I will modify the previous argument. Again I will perform a $\kappa + 1$ reverse Easton iteration $\mathbb{P}_\kappa * \mathbb{Q}_\kappa$, where at each stage γ I perform one of three kinds of forcing. First, the Laver function may instruct us, as before, to perform the fragility forcing. Second, the Laver function may instruct us to perform $< \gamma$ -directed closed forcing \mathbb{Q}_γ , and we will oblige, *provided* that \mathbb{Q}_γ collapses γ^+ . Finally, third, the Laver function may instruct us to perform both of the previous two types of forcing. Let $G * g$ be V -generic for $\mathbb{P}_\kappa * \mathbb{Q}_\kappa$, where \mathbb{Q}_κ is as previously the stage κ fragility forcing using the κ -data packet $\langle a_\kappa, d_\kappa \rangle$, and let me show that $V[G][g]$ has the properties that we seek.

First, I will prove that κ is supercompact in $V[G][g]$. Fix any λ , and select $\theta \gg \lambda$, and a θ -supercompact embedding $j : V \rightarrow M$ such that $j(\ell)(\kappa)$ tells us to perform just the fragility forcing, using $\langle a_\kappa, d_\kappa \rangle$, and that $\text{dom}(j(\ell)) \cap (\kappa, \theta] = \emptyset$. Thus, $j(\mathbb{P}) = \mathbb{P} * \mathbb{Q}_\kappa * \mathbb{P}_{\text{tail}}$, where \mathbb{P}_{tail} is $\leq \theta$ -closed. As usual, force over the tail, and lift the embedding to $j : V[G] \rightarrow M[j(G)]$, where $j(G) = G * g * G_{\text{tail}}$. Now use the master condition argument from Possibility One to lift through the \mathbb{Q}_κ forcing. This gives $j : V[G][g] \rightarrow M[j(G)][j(g)]$ in $V[G][g][G_{\text{tail}}][j(g)]$. The measure on $P_\kappa \lambda$ germinated by the seed $j \restriction \lambda$ must lie in $V[G][g]$, so κ is λ -supercompact there.

Next, I will establish that κ is fragile in $V[G][g]$. This is nearly identical to the corresponding argument in Possibility One. Suppose that κ is measurable in

$V[G][g][H]$, where $H \subseteq \mathbb{Q}$ adds a new subset $A \subseteq \kappa$, but preserves $\kappa^{<\kappa}$ and κ^+ . Thus, there is an embedding $j : V[G][g][H] \rightarrow M[j(G)][j(g)][j(H)]$. Moreover, since κ is an inaccessible cluster point of the club sets in $j(g)$, namely $j(C_\kappa^\alpha)$, it follows that the α^{th} generic club added at stage κ is $j(C_\kappa^\alpha) \cap \kappa = C_\kappa^\alpha$. Thus, since as before κ^+ is preserved, $j(G) = G * g * G_{\text{tail}}$ for some G_{tail} . Furthermore, by the reflection property of the stage $j(\kappa)$ data packets to the stage κ data packets, I also know $P(\kappa)^M = P(\kappa)^V$, since both are enumerated by d_κ . Also, κ^+ is the same in all the models since it is not collapsed by G , g , or H . Thus there was no supplementary Laver forcing at stage κ in $j(\mathbb{P})$. So I can proceed as before, and obtain the contradiction involving \dot{A} .

Finally, let me prove that the supercompactness of κ is preserved over $V[G][g]$ by any $<\kappa$ -directed closed poset which collapses κ^+ . Suppose that $H \subseteq \mathbb{Q}$ is generic, where \mathbb{Q} is $<\kappa$ -directed closed and collapses κ^+ . Fix any λ and pick $\theta \gg \lambda$. Fix $j : V \rightarrow M$ a θ -supercompact embedding such that $j(\ell)(\kappa)$ instructs us to perform the fragility forcing with $\langle a_\kappa, d_\kappa \rangle$, followed by \mathbb{Q} . Also I will need that $\text{dom}(j(\ell)) \cap (\kappa, \theta] = \emptyset$. Let me proceed to lift j to the forcing extension. The stage κ forcing in $j(\mathbb{P}_\kappa)$ is no problem since I have $g * H$. Force over the tail to get G_{tail} , and lift to $j : V[G] \rightarrow M[j(G)]$, where $j(G) = G * g * H * G_{\text{tail}}$. Now use the master condition below $j \restriction g$ and force to add $j(g)$, lifting to $j : V[G][g] \rightarrow M[j(G)][j(g)]$. Similarly, using the directed closure, I can find a master condition below $j \restriction H$ and lift to $j : V[G][g][H] \rightarrow M[j(G)][j(g)][j(H)]$. Finally, use $j \restriction \lambda$ as a seed and observe that κ is still λ -supercompact in $V[G][g][H]$. The measure could not have been added by $G_{\text{tail}} * j(g) * j(H)$, because of closure, and therefore lies in $V[G][g][H]$. This completes the proof. $\square_{\text{Possibility Two}}$

Possibility Three. *There is a forcing extension in which κ is superdestructible, but not fragile.*

Proof: First force with the Laver preparation to $V[G]$ where κ is indestructible. Then, perform any small forcing $g \subseteq \mathbb{Q}$, such as adding a single Cohen real. It follows by the Superdestruction Theorem of [Ham97b] that κ is superdestructible in $V[G][g]$, and I will now show that κ is not fragile there. For this, it suffices to show that forcing over $\mathbb{R} = \text{Add}(\kappa, 1)^{V[G]}$ will preserve the supercompactness of κ over $V[G][g]$. Certainly forcing with \mathbb{R} over $V[G]$ preserves the supercompactness of κ , since κ is indestructible in $V[G]$. Also, the further small forcing \mathbb{Q} preserves supercompactness again. So over $V[G]$, the forcing $\mathbb{R} \times \mathbb{Q}$ preserves supercompactness. Thus, by rearranging the order of the forcing, $\mathbb{Q} \times \mathbb{R}$ also preserves supercompactness. Thus,

\mathbb{R} preserves supercompactness over $V[G][g]$, as desired. $\square_{\text{Possibility Three}}$

Possibility Four. *There is a forcing extension in which κ is neither fragile nor superdestructible.*

Proof: The Laver preparation makes κ indestructible, and therefore neither fragile nor superdestructible. $\square_{\text{Possibility Four}}$

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